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Advances in Thermal Analysis of Heterogeneous Materials with Hybrid Finite Element Methods

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ABSTRACT

This paper presents an overview on applications of hybrid finite element method to thermal analysis of heterogeneous materials. Recent developments on the hybrid fundamental solution based finite element model (FEM) of heat transfer in nonlinear functionally graded materials (FGMs) and composite materials are described. Formulations for all cases are derived by means of modified variational functional and fundamental solutions. Generation of elemental stiffness equations from the modified variational principle is also discussed. Finally, a brief summary of the approach is provided.

Keywords: Finite Element Method, Fundamental Solution, Functionally Graded Material, Composite Materials

I. INTRODUCTION

Heterogeneous materials such as functionally graded materials (FGMs) [1-4] and composites materials in general [5-9] have been widely used in the space engineering to achieve some special purposes. For instance, FGMs, which may be characterized by a spatial gradation variation in composition and structure, was first considered in 1984 by material scientists in the Sendai area in Japan during a space plane project for the special purpose of preparing materials with high thermal barrier capability [10]

Recently, two effective numerical methods have been developed for analysing mechanical performance of heterogeneous materials [11-13]. The first is the so-called hybrid Trefftz FEM (or T-Trefftz method) [14, 15]. Unlike in the conventional FEM, the T-Trefftz method couples the advantages of conventional FEM [16-19] and BEM [20-22]. In contrast to the standard FEM, the T-Trefftz method is based on a hybrid method which includes the use of an independent auxiliary inter-element frame field defined on each element boundary and an independent internal field chosen so as to a prior satisfy the homogeneous governing differential equations by means of a suitable truncated T-complete function set of homogeneous solutions. Since 1970s, T-Trefftz model has been considerably improved and has now become a highly efficient computational tool for the solution of complex boundary value problems. It has been applied to potential problems [23-26], two-dimensional elastics [27, 28], elastoplasticity [29, 30], fracture mechanics [31-33], micromechanics analysis [34, 35], problem with holes [36, 37], heat conduction [38-40], thin plate bending [41-44], thick or moderately thick plates [45-49], three-dimensional problems [50], piezoelectric materials [51-55], and contact problems [56-58].

On the other hand, the hybrid FEM based on the fundamental solution (F-Trefftz method for short) was initiated in 2008 [15, 59] and has now become a very popular and powerful computational methods in mechanical engineering. The F-Trefftz method is significantly different from the T-Trefftz method discussed above. In this method, a linear combination of the fundamental solution at different points is used to approximate the field variable within the element. The independent frame field defined along the element boundary and the newly developed variational functional are employed to guarantee the inter-element continuity, generate the final stiffness equation and establish linkage between the boundary frame field and internal field in the element. This review will focus on the F-Trefftz finite element method.

The F-Trefftz finite element method, newly developed recently [15, 59], has gradually become popular in the field of mechanical and physical engineering since it is initiated in 2008 [15, 60, 61]. It has been applied to potential problems [25, 62-64], plane elasticity [28, 65, 66], composites [67-70], piezoelectric materials [71-73], three-dimensional problems [74], functionally graded materials [75-77], bioheat transfer problems [78-82], thermal elastic problems [83], hole problems [84, 85], heat conduction problems [59, 86], micromechanics problems [34, 35], and anisotropic elastic problems [87-89].

Following this introduction, the present review consists of 3 sections. F-Trefftz FEM for nonlinear heat transfer in FGMs is described in Section 2. It describes in detail the method of deriving element stiffness equations. Section 3 focuses on the essentials of F-Trefftz elements for composites. Finally, a brief summary of the developments of the Treffz methods is provided.

II. METHODS AND MATERIAL

A. F-Trefftz method for Nonlinear FGMs

1. Basic formulations

Consider a two-dimensional (2D) heat conduction problem defined in an anisotropic inhomogeneous media [90]:

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial X_i}(\tilde{K}_{ij}(\mathbf{X},u)\frac{\partial u(\mathbf{X})}{\partial X_j})=0 \qquad \forall \mathbf{X}\in\Omega \qquad (1)$$

For an inhomogeneous nonlinear functionally graded material, we assume the thermal conductivity varies exponentially with position vector and also be a function of temperature, that is

$$\tilde{K}_{ij}(\mathbf{X},u)=\alpha(u)K_{ij}\exp(2\boldsymbol{\beta}\cdot\mathbf{X}) \qquad (2)$$

where $\alpha(u)>0$ is a function of temperature which may be different for different materials, the vector $\boldsymbol{\beta}=(\beta_1,\beta_2)$ is a dimensionless graded parameter and matrix $\mathbf{K}=[K_{ij}]_{1\le i,j\le 2}$ is a symmetric, positive-definite constant matrix ($K_{12}=K_{21}, \det K=K_{11}K_{22}-K_{12}{}^2>0$). The boundary conditions are as follows:

—Dirichlet boundary condition

$$u=\bar{u} \qquad \text{on } \Gamma_u \qquad (3)$$

—Neumann boundary condition

$$q=-\sum_{i,j=1}^{2}\tilde{K}_{ij}\frac{\partial u}{\partial X_j}n_i=\bar{q} \qquad \text{on } \Gamma_q \qquad (4)$$

where $\tilde{K}_{ij}$ denotes the thermal conductivity which is the function of spatial variable $\mathbf{X}$ and unknown temperature field u . q represents the boundary heat flux. n_j is the direction cosine of the unit outward normal vector $\mathbf{n}$ to the boundary $\Gamma=\Gamma_u\cup\Gamma_q$. $\bar{u}$ and $\bar{q}$ are specified functions on the related boundaries, respectively.

2. Kirchhoff transformation and iterative Method

In general, many approaches have been reported for solving nonlinear problems [91-95]. In this work, two methods are employed to deal with the nonlinear term $\alpha(u)$, one is Kirchhoff transformation [96] and another is the iterative method.

(1) Kirchhoff Transformation

$$\Psi(u)=\psi(u(\mathbf{X}))=\int\alpha(u)du \qquad (5)$$

Making use of Eq.(5), Eq.(1) reduces to

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial X_i}(K_{ij}^*(\mathbf{X})\frac{\partial\Psi(\mathbf{X})}{\partial X_j})=0 \quad \forall\mathbf{X}\in\Omega \qquad (6)$$

where

$$K_{ij}^*(\mathbf{X})=K_{ij}\exp(2\boldsymbol{\beta}\cdot\mathbf{X}) \qquad (7)$$

Substituting Eq.(7) into Eq.(6) yields

$$\left[\sum_{i,j=1}^{2}K_{ij}\frac{\partial^2\Psi(\mathbf{X})}{\partial X_i\partial X_j}+2\boldsymbol{\beta}\cdot(\mathbf{K}\nabla\Psi(\mathbf{X}))\right]\exp(2\boldsymbol{\beta}\cdot\mathbf{X})=0 \quad (8)$$

where

$$u=\psi^{-1}(\Psi) \qquad (9)$$

It should be mentioned that the inverse of Ψ in Eq.(9) exists since $\alpha(u)>0$.

The fundamental solution to Eq.(8) in two dimensions can be expressed as [96]

$$N(\mathbf{X},\mathbf{X}_s)=-\frac{K_0(\kappa R)}{2\pi\sqrt{\det\mathbf{K}}}\exp\{-\boldsymbol{\beta}\cdot(\mathbf{X}+\mathbf{X}_s)\} \quad (10)$$

where $\kappa = \sqrt{\boldsymbol{\beta} \cdot \mathbf{K}\boldsymbol{\beta}}$, R is the geodesic distance defined as $R = R(\mathbf{X}, \mathbf{X}_s) = \sqrt{\mathbf{r} \cdot \mathbf{K}^{-1}\mathbf{r}}$ and $\mathbf{r} = \mathbf{X} - \mathbf{X}_s$ in which $\mathbf{X}$ and $\mathbf{X}_s$ denote observing field point and source point in the infinite domain, respectively. K_0 is the modified Bessel function of the second kind of zero order. For isotropic materials, $K_{12} = K_{21} = 0$, $K_{11} = K_{22} = k_0 > 0$, then the fundamental solution given by (9) reduces to

$$N(\mathbf{X}, \mathbf{X}_s) = -\frac{K_0(\kappa R)}{2\pi k_0} \exp\{-\boldsymbol{\beta} \cdot (\mathbf{X} + \mathbf{X}_s)\} \quad (11)$$

which agrees with the result in [97].

Under the Kirchhoff transformation, the boundary conditions (3)-(4) are transformed into the corresponding boundary conditions in terms of Ψ.

$$\Psi = \psi(\bar{u}) \quad \text{on } \Gamma_u \quad (12)$$

$$p = -\sum_{i,j=1}^{2} K_{ij}^{*} \frac{\partial \Psi}{\partial X_j} n_i = -\sum_{i,j=1}^{2} \tilde{K}_{ij} \frac{\partial u}{\partial X_j} n_i = q = \bar{q} \text{ on } \Gamma_q \quad (13)$$

Therefore, by Kirchhoff transformation, the original nonlinear heat conduction equation (1), in which the heat conductivity is a function of coordinate X and unknown function u, can be transformed into the linear equation (6) in which the heat conductivity is just a function of coordinate X. At the same time, the field variable becomes Ψ in Eq.(6), rather than u in Eq.(1). The boundary conditions (3)-(4) are correspondingly transformed into Eqs.(12)-(13). Once Ψ is determined, the temperature solution u can be found by the reversion of transformation (9), i.e. $u = \psi^{-1}(\Psi)$.

(2) Iterative Method

Since the heat conductivity depends on the unknown function u, an iterative procedure is employed for determining the temperature distribution. The algorithm is given as follows:

1. Assume an initial temperature u^0.
2. Calculate the heat conductivity in Eq.(2) using u^0.
3. Solve the boundary value problem defined by Eqs.(1)-(4) for the temperature u
4. Define the convergent criterion $|u - u^0| < \delta$ (δ=10^{-6} in our analysis). If the criterion is satisfied, output the result and terminate the process. If not satisfied, go to next step.
5. Update u^0 with u
6. Go to step 2.

3. Generation of graded element

In this section, an element formulation is presented to deal with materials with continuous variation of physical properties. Such an element model is usually known as a hybrid graded element which can be used for solving the boundary value problem (BVP) defined in Eqs.(6) and (12)-(13). As was done in conventional FEM, the solution domain is divided into sub-domains or elements. For a particular element, say element e, its domain is denoted by Ω_e and bounded by Γ_e. Since a nonconforming function is used for modeling intra-element field, additional continuities are usually required over the common boundary Γ_{Ief} between any two adjacent elements 'e' and 'f' [59]

$$\left.\begin{aligned} \Psi_e &= \Psi_f \quad \text{(conformity)} \\ p_e + p_f &= 0 \quad \text{(reciprocity)} \end{aligned}\right\} \text{ on } \Gamma_{Ief} = \Gamma_e \cap \Gamma_f \quad (14)$$

in the proposed hybrid FE approach.

Non-Conforming Intra-Element Field

For a particular element, say element e, which occupies sub-domain Ω_e, the field variable within the element is extracted from a linear combination of fundamental solutions centered at different source points that[59]

$$\Psi_e(\mathbf{x}) = \sum_{j=1}^{n_s} N_e(\mathbf{x}, \mathbf{y}_j) c_{ej} = \mathbf{N}_e(\mathbf{x}) \mathbf{c}_e \quad \forall \mathbf{x} \in \Omega_e, \mathbf{y}_j \notin \Omega_e \quad (15)$$

where c_{ej} is undetermined coefficients and n_s is the number of virtual sources outside the element e. $N_e(\mathbf{x}, \mathbf{y}_j)$ is the required fundamental solution expressed in terms of local element coordinates (x_1, x_2), instead of global coordinates (X_1, X_2).

The fundamental solution for FGM (N_e in Eq.(15)) is used to approximate the intra-element field in FGM. It is well known that the fundamental solution represents the filed generated by a concentrated unit source acting at a point, so the smooth variation of material properties throughout an element can be achieved by this inherent property, instead of the stepwise constant approximation, which has been frequently used in the conventional FEM.

Note that the thermal conductivity in Eq. (7) is defined in the global coordinate system. When contriving the intra-element field for each element, this formulation has to be transferred into local element coordinate system defined at the center of the element, the graded matrix $\mathbf{K}^*$ in Eq. (7) can, then, be expressed by

$$\mathbf{K}_e^*(\mathbf{x}) = \mathbf{K}_C \exp(2\boldsymbol{\beta}\cdot\mathbf{x}) \tag{16}$$

for a particular element e, where $\mathbf{K}_C$ denotes the value of conductivity at the centroid of each element and can be calculated as follows:

$$\mathbf{K}_C = \mathbf{K}\exp(2\boldsymbol{\beta}\cdot\mathbf{X}_c) \tag{17}$$

where $\mathbf{X}_c$ is the global coordinates of the element centroid.

Accordingly, the matrix $\mathbf{K}_C$ is used to replace $\mathbf{K}$ (see Eq.(90)) in the formulation of fundamental solution for FGM and to construct intra-element field in the coordinate system local to element.

In practice, the generation of virtual sources is usually done by means of the following formulation employed in the MFS [96]

$$\mathbf{y} = \mathbf{x}_b + \mu\left(\mathbf{x}_b - \mathbf{x}_c\right) \tag{18}$$

where μ is a dimensionless coefficient (μ =2.5 in our analysis[98]), $\mathbf{x}_b$ and $\mathbf{x}_c$ are, respectively, boundary point and geometrical centroid of the element. For a particular element shown in Fig. 2, we can use the nodes of element to generate related source points.

+The corresponding normal heat flux on Γ_e is given by

$$p_e = -\mathbf{K}_e^* \frac{\partial \Psi_e}{\partial X_j} n_i = \mathbf{Q}_e \mathbf{c}_e \tag{19}$$

where

$$\mathbf{Q}_e = -\mathbf{K}_e^* \frac{\partial \mathbf{N}_e}{\partial X_j} n_i = -\mathbf{A}\mathbf{K}_e^*\mathbf{T}_e \tag{20}$$

with

$$\mathbf{T}_e = \begin{bmatrix}\mathbf{N}_{e,1} & \mathbf{N}_{e,2}\end{bmatrix}^{\mathrm{T}} \qquad \mathbf{A} = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \tag{21}$$

Auxiliary conforming frame field

In order to enforce the conformity on the field variable u, for instance, $\Psi_e = \Psi_f$ on $\Gamma_e \cap \Gamma_f$ of any two neighboring elements e and f, an auxiliary inter-element frame field $\tilde{\Psi}$ is used and expressed in terms of nodal degrees of freedom (DOF), $\mathbf{d}$, as used in the conventional finite elements as

$$\tilde{\Psi}_e(\mathbf{x}) = \tilde{\mathbf{N}}_e(\mathbf{x})\mathbf{d}_e \tag{22}$$

which is independently assumed along the element boundary, where $\tilde{\mathbf{N}}_e$ represents the conventional FE interpolating functions. For example, a simple interpolation of the frame field on the side with three nodes of a particular element can be given in the form

$$\tilde{\Psi} = \tilde{N}_1\Psi_1 + \tilde{N}_2\Psi_2 + \tilde{N}_3\Psi_3 \tag{23}$$

where $\tilde{N}_i$ ($i = 1,2,3$) stands for shape functions in terms of natural coordinate ξ defined in Fig. 3.

4 Modified variational principle and stiffness equation

Modified Variational Functional [90]

For the boundary value problem defined in Eqs.(6) and (12)-(13), since the stationary conditions of the traditional potential or complementary variational functional can't guarantee the satisfaction of inter-element continuity condition required in the proposed HFS-FE model, a modified potential functional is developed as follows [15]

$$\begin{aligned}\Pi_m = \sum_e \Pi_{me} = \sum_e [-\int_{\Omega_e} \frac{1}{2} K_{ij}^* \Psi_{,i}\Psi_{,j} \mathrm{d}\Omega \\ -\int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} \left(\tilde{\Psi} - \Psi\right) p \mathrm{d}\Gamma]\end{aligned} \tag{24}$$

in which the governing equation (6) is assumed to be satisfied, a priori, in deriving the HFS-FE model (For convenience, the repeated subscript indices stand for summation convention). The boundary Γ_e of a particular element consists of the following parts

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{qe} \cup \Gamma_{Ie} \tag{25}$$

where Γ_{Ie} represents the inter-element boundary of the element 'e'[15].

The stationary condition of the functional (104) can lead to the governing equation (Euler equation), boundary conditions and continuity conditions, details of the derivation can refer to Ref. [15].

Stiffness Equation

Having independently defined the intra-element field and frame field in a particular element[15], the next step is to generate the element stiffness equation through a variational approach and to establish a linkage between the two independent fields.

The variational functional Π_e corresponding to a particular element e of the present problem can be written as

$$\Pi_{me} = -\frac{1}{2}\int_{\Omega_e} K_{ij}^{*}\Psi_{,i}\Psi_{,j}\mathrm{d}\Omega - \int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\left(\tilde{\Psi}-\Psi\right)\mathrm{d}\Gamma \quad (26)$$

Appling the Gauss theorem to the above functional, we have the following functional for the F-Trefftz model

$$\Pi_{me} = \frac{1}{2}\left[\int_{\Gamma_e} p\Psi\mathrm{d}\Gamma + \int_{\Omega_e} \Psi(K_{ij}^{*}u_{,i})_{,j}\mathrm{d}\Omega\right] - \int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\left(\tilde{\Psi}-\Psi\right)\mathrm{d}\Gamma \quad (27)$$

Considering the governing equation (6), we finally have the functional defined on the element boundary only

$$\Pi_{me} = -\frac{1}{2}\int_{\Gamma_e} p\Psi\mathrm{d}\Gamma - \int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\tilde{\Psi}\mathrm{d}\Gamma \quad (28)$$

which yields by substituting Eqs (15), (19) and (22) into the functional (28)

$$\Pi_e = -\frac{1}{2}\mathbf{c}_e^{\mathrm{T}}\mathbf{H}_e\mathbf{c}_e - \mathbf{d}_e^{\mathrm{T}}\mathbf{g}_e + \mathbf{c}_e^{\mathrm{T}}\mathbf{G}_e\mathbf{d}_e \quad (29)$$

with

$$\mathbf{H}_e = \int_{\Gamma_e}\mathbf{Q}_e^{\mathrm{T}}\mathbf{N}_e\mathrm{d}\Gamma,\ \mathbf{G}_e = \int_{\Gamma_e}\mathbf{Q}_e^{\mathrm{T}}\tilde{\mathbf{N}}_e\mathrm{d}\Gamma,\ \mathbf{g}_e = \int_{\Gamma_{qe}}\tilde{\mathbf{N}}_e^{\mathrm{T}}\bar{\mathrm{q}}\mathrm{d}\Gamma \quad (30)$$

B. F-Trefftz method in composite materials

1 Basic equations of composite materials

The heat conduction in composites which usually include two or more material constituents is more complicated than that in homogeneous materials. Here, for the sake of convenience, the heat conduction in fibre-reinforced composites is reviewed, as shown in Fig. 1. Assume that $\mathbf{x}=(x_1,x_2)$ denotes the spatial coordinates, Ω_m and Ω_f are the matrix and fiber domains, respectively, and u_m and u_f are the temperature fields in the corresponding domains. If both the matrix and the fiber are isotropic, then, the heat transfer equations in fibre-reinforced composites can be written as follows.

(1) the governing equations for the fiber and matrix material phases:

$$\frac{\partial^2 u_m}{\partial x_1^2}+\frac{\partial^2 u_m}{\partial x_2^2}=0, \qquad \mathbf{x}\in\Omega_m$$
$$\frac{\partial^2 u_f}{\partial x_1^2}+\frac{\partial^2 u_f}{\partial x_2^2}=0, \qquad \mathbf{x}\in\Omega_f \quad (31)$$

(2) the continuity conditions at the interface Γ_{mf} between the fiber and the matrix

$$u_m = u_f, \qquad k_m\frac{\partial u_m}{\partial n} = k_f\frac{\partial u_f}{\partial n} \quad (32)$$

where n is the unit direction normal to the fiber/matrix interface, and k_m and k_f are the thermal conductivities of the matrix and fiber materials.

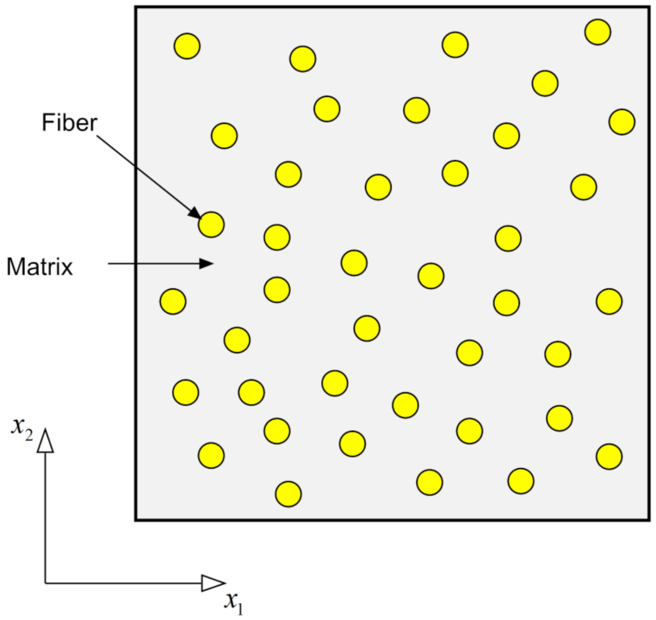


Figure 1. Schematic of fiber-reinforced composites

Besides, according to the Fourier's law of heat transfer in isotropic media, one has the following relationship between the temperature variable u and the heat flux component q_i:

$$q_{mi} = -k_m \frac{\partial u_m}{\partial x_i} \qquad (i=1,2)$$
$$q_{fi} = -k_f \frac{\partial u_f}{\partial x_i} \qquad (i=1,2) \tag{33}$$

Finally, the boundary conditions of the problem defined along the outer boundary of the matrix domain is given by

$$Bu_m(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega_{m1} \tag{34}$$

where B stands for a boundary operator and $f(\mathbf{x})$ is a given function along the boundary. $\partial\Omega_{m1} = \partial\Omega_m \setminus \Gamma_{mf}$ is the outer boundary of the matrix domain.

2 Formulation for F-Trefftz method

According to the feature that arbitrarily shaped polygonal hybrid finite element can be constructed in the HFS-FEM, the special n-sided fiber/matrix elements can be developed for such analysis, as described below.

2.1 Special fundamental solutions

Fundamental solutions play an important role in the derivation of the special n-sided fiber/matrix elements. With the well-developed F-Trefftz finite element method for two-dimensional heat conduction problems in fiber-reinforced composites, the temperature response in an infinite matrix region containing a centered circular fiber is needed under the unit heat source in the matrix. In Fig. 2, a unit heat source is applied at the source point z_0 in the infinite matrix Ω_m, then the temperature responses G_m and G_f at any field point z in matrix and fiber regions are respectively obtained as [99]

$$\begin{cases} G_m = -\dfrac{1}{2\pi k_m}\left[\begin{array}{l}\mathrm{Re}[\ln(z-z_0)] \\ +\dfrac{k_m-k_f}{k_m+k_f}\mathrm{Re}[\ln(\dfrac{R^2}{z}-\bar{z}_0)]\end{array}\right] & z\in\Omega_m \\ G_f = -\dfrac{1}{(k_m+k_f)\pi}\mathrm{Re}[\ln(z-z_0)] & z\in\Omega_f \end{cases} \tag{35}$$

where R is the radius of the fiber, $z = x_1 + x_2 i$ is a complex number defined in a local coordinate system $\mathbf{x} = (x_1, x_2)$ with its origin coincident with the fiber center, and $i=\sqrt{-1}$ denotes the unit imaginary number.

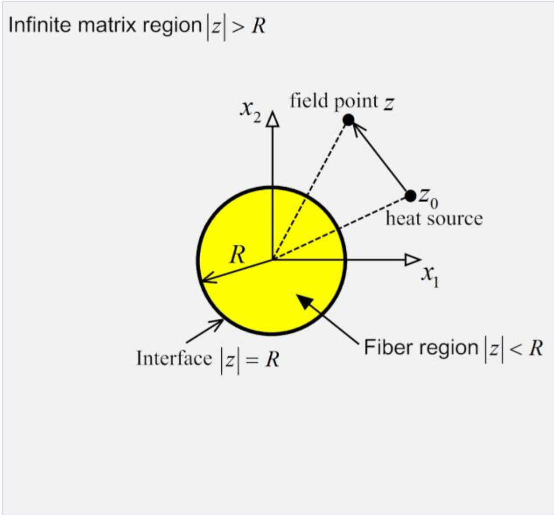
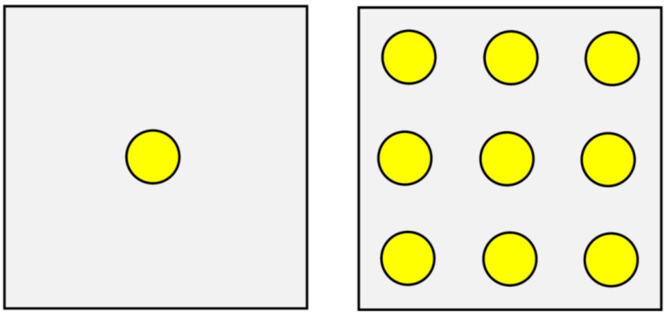


Figure 2. Special fundamental solutions for heat conduction problems in fiber-reinforced composites

2.2 Special n-sided fiber/matrix elements

By means of the special fundamental solutions described above, special n-sided fiber/matrix elements can be constructed. Here, one and nine fibers embedded in the unit square matrix region (see Fig. 3) are considered to demonstrate the performance of the present special n-sided fiber/matrix elements. A uniform heat flux of 100.0 is applied on the left side of the square matrix region, while its opposite side is maintained at a temperature of 0. All other edges of the square are assumed to be insulated.

For simplicity, the thermal conductivities in consistent units are assumed for the matrix and the fiber, that is, $k_f = 20k_m = 20$. Figs. 4 and 5 show respectively the temperature distribution along the middle line $x_2 = 0.5$ for the cases of one and nine fibers. Also, the numerical results from the conventional finite elements (ABAQUS) are provided as reference values for comparison. To obtain steady results, the conventional finite element mesh is chosen to be refined enough. It's found that there are good agreements between results from the present F-Trefftz finite element method and ABAQUS for both cases. However, very few elements are employed in the present F-Trefftz finite element method and mesh discretization inside the fiber domain is avoided.



(a) single fiber (b) nine fibers

Figure 3. Different fiber numbers in the matrix

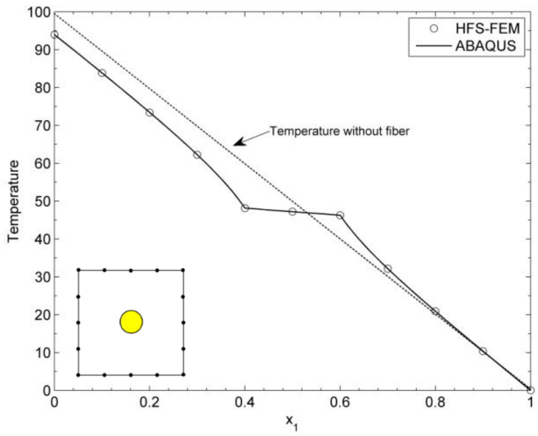


Figure 4. Temperature variation along the line x2=0.5 for one fiber case

III. CONCLUSION

On the basis of the preceding discussion, the following conclusions can be drawn. This review reports recent developments on applications of F-Trefftz finite element method to heterogeneous materials and structures. It proved to be a powerful computational tool in modeling materials and structures with inhomogeneous properties. However, there are still many possible extensions and areas in need of further development in the future. Among those developments one could list the following:

1. Development of efficient F-Trefftz FE-BEM schemes for complex engineering structures containing heterogeneous materials and the related general purpose computer codes with preprocessing and postprocessing capabilities.
2. Generation of various special-purpose elements to effectively handle singularities attributable to local geometrical or load effects (holes, cracks, inclusions, interface, corner and load singularities). The special-purpose functions warrant that excellent results are obtained at minimal computational cost and without local mesh refinement
3. Development of F-Trefftz FE in conjunction with a topology optimization scheme to contribute to microstructure design.
4. Extension of the F-Trefftz FEM to elastodynamics and fracture mechanics of FGMs.

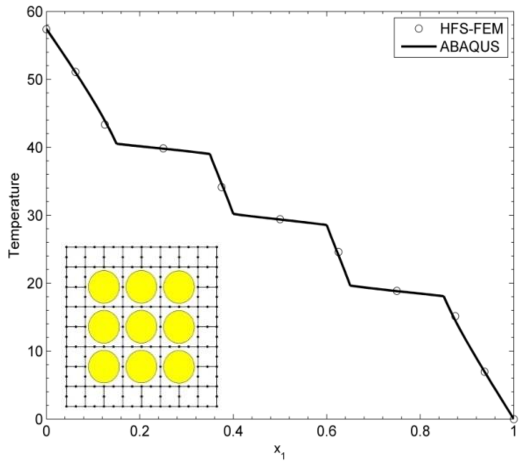


Figure 5. Temperature variation along the line x2=0.5 for nine fiber case

IV. REFERENCES

[1] J. Reddy, Analysis of functionally graded plates, International Journal for Numerical Methods in Engineering, 47(1-3) (2000) 663-684.

[2] E. Carrera, S. Brischetto, A. Robaldo, Variable kinematic model for the analysis of functionally graded material plates, AIAA journal, 46(1) (2008) 194-203.

[3] J.-H. Kim, G. Paulino, Isoparametric graded finite elements for nonhomogeneous isotropic and orthotropic materials, Journal of Applied Mechanics, 69(4) (2002) 502-514.

[4] H. Wang, Q.H. Qin, Y. Kang, A meshless model for transient heat conduction in functionally graded materials, Computational Mechanics, 38(1) (2006) 51-60.

[5] Y. Jiang, J. Ou, Z. Zhang, Q.-H. Qin, Preparation of magnetic and bioactive calcium zinc iron silicon oxide composite for hyperthermia treatment of bone cancer and repair of bone defects, Journal of Materials Science: Materials in Medicine, 22(3) (2011) 721-729.

[6] H.Y. Liu, Q.H. Qin, Y.W. Mai, Theoretical model of piezoelectric fibre pull-out, International Journal of Solids and Structures, 40(20) (2003) 5511-5519.

[7] W. Qiu, Y. Kang, Q. Sun, Q.H. Qin, Y. Lin, Stress analysis and geometrical configuration selection for multilayer piezoelectric displacement actuator, Acta Mechanica Solida Sinica, 17(4) (2004) 323-329.

[8] C.-Y. Lee, Q.H. Qin, G. Walpole, Numerical modeling on electric response of fibre-orientation of composites with piezoelectricity, International Journal of Research and Reviews in Applied Science, 16(3) (2013) 377-386.

[9] V. Kompiš, Z. Murčinková, S. Rjasanow, R. Grzhibovskis, Q.-H. Qin, Computational Simulation Methods for Composites Reinforced by Fibres, Computational Structural Engineering: Proceedings of the International Symposium on Computational Structural Engineering, Held in Shanghai, China, June 22-24, 2009, (2009) 63.

[10] M. Koizumi, FGM activities in Japan, Composites Part B: Engineering, 28(1) (1997) 1-4.

[11] H. Wang, Q.-H. Qin, Y. Xiao, Special n-sided Voronoi fiber/matrix elements for clustering thermal effect in natural-hemp-fiber-filled cement composites, International Journal of Heat and Mass Transfer, 92 (2016) 228-235.

[12] Q.H. Qin, The Trefftz finite and boundary element method, WIT Press, Southampton, 2000.

[13] Q.H. Qin, Trefftz finite element method and its applications, Applied Mechanics Reviews, 58(5) (2005) 316-337.

[14] Z.-J. Fu, Q.-H. Qin, W. Chen, Hybrid-Trefftz finite element method for heat conduction in nonlinear functionally graded materials, Engineering Computations, 28(5) (2011) 578-599.

[15] Q.H. Qin, H. Wang, Matlab and C programming for Trefftz finite element methods, New York: CRC Press, 2008.

[16] Q.H. Qin, C.X. Mao, Coupled torsional-flexural vibration of shaft systems in mechanical engineering—I. Finite element model, Computers & Structures, 58(4) (1996) 835-843.

[17] H.C. Martin, G.F. Carey, Introduction to finite element analysis: Theory and application, McGraw-Hill New York, 1973.

[18] H. Wang, Y. Xiao, Q.H. Qin, 2D hierarchical heat transfer computational model of natural ber bundle reinforced composite, Scientia Iranica, Transactions B: Mechanical Engineering, 23(1) (2016) 268-276.

[19] K.Z. Ding, Q.H. Qin, M. Cardew-Hall, A New Integration Algorithm for the Finite Element Analysis of Elastic-Plastic Problems, in: Proc. of 9th International Conference on Inspection, Appraisal, Repairs & Maintenance of Structures, Fuzhou, China, 20-21 October, CI-Premier PTY LTD, ISBN: 981-05-3548-1, 2005, pp. 209-216.

[20] Q.H. Qin, Y.W. Mai, BEM for crack-hole problems in thermopiezoelectric materials, Engineering Fracture Mechanics, 69(5) (2002) 577-588.

[21] Q.H. Qin, Y. Huang, BEM of postbuckling analysis of thin plates, Applied Mathematical Modelling, 14(10) (1990) 544-548.

[22] Q.H. Qin, Nonlinear analysis of Reissner plates on an elastic foundation by the BEM, International journal of solids and structures, 30(22) (1993) 3101-3111.

[23] W. Chen, Z.-J. Fu, Q.-H. Qin, Boundary particle method with high-order Trefftz functions, Computers, Materials & Continua (CMC), 13(3) (2010) 201-217.

[24] H. Wang, Q.H. Qin, D. Arounsavat, Application of hybrid Trefftz finite element method to non-linear problems of minimal surface, International Journal for Numerical Methods in Engineering, 69(6) (2007) 1262-1277.

[25] H. Wang, Q.-H. Qin, X.-P. Liang, Solving the nonlinear Poisson-type problems with F-Trefftz hybrid finite element model, Engineering Analysis with Boundary Elements, 36(1) (2012) 39-46.

[26] L.-L. Cao, Q.-H. Qin, N. Zhao, A new RBF-Trefftz meshless method for partial differential equations, IOP Conference Series: Materials Science and Engineering, 10 (2010) 012217.

[27] Q.H. Qin, Dual variational formulation for Trefftz finite element method of elastic materials, Mechanics Research Communications, 31(3) (2004) 321-330.

[28] H. Wang, Q.-H. Qin, Numerical implementation of local effects due to two-dimensional discontinuous loads using special elements based on boundary integrals, Engineering Analysis with Boundary Elements, 36(12) (2012) 1733-1745.

[29] Q.H. Qin, Formulation of hybrid Trefftz finite element method for elastoplasticity, Applied Mathematical Modelling, 29(3) (2005) 235-252.

[30] Q.-H. Qin, Trefftz plane elements of elastoplasticity with p-extension capabilities, Journal of Mechanical Engineering, 56(1) (2005) 40-59.

[31] Y. Cui, Q.H. Qin, Fracture analysis of mode III problems by Trefftz finite element approach, in: WCCM VI in conjunction with APCOM, 2004, pp. v1-p380.

[32] Y.-h. Cui, Q.H. Qin, J.-S. Wang, Application of HT finite element method to multiple crack problems of Mode I, II and III, Chinese Journal of Engineering Mechanics, 23(3) (2006) 104-110.

[33] Y. Cui, J. Wang, M. Dhanasek, Q.H. Qin, Mode III fracture analysis by Trefftz boundary element method, Acta Mechanica Sinica, 23(2) (2007) 173-181.

[34] C. Cao, Q.H. Qin, A. Yu, Micromechanical Analysis of Heterogeneous Composites using Hybrid Trefftz FEM and Hybrid Fundamental Solution Based FEM, Journal of Mechanics, 29(4) (2013) 661-674.

[35] C. Cao, A. Yu, Q.-H. Qin, Evaluation of effective thermal conductivity of fiber-reinforced composites by boundary integral based finite element method, International Journal of Architecture, Engineering and Construction, 1(1) (2012) 14-29.

[36] M. Dhanasekar, J. Han, Q.H. Qin, A hybrid-Trefftz element containing an elliptic hole, Finite Elements in Analysis and Design, 42(14) (2006) 1314-1323.

[37] Q.-H. Qin, X.-Q. He, Special elliptic hole elements of Trefftz FEM in stress concentration analysis, Journal of Mechanics and MEMS, 1(2) (2009) 335-348.

[38] J. Jirousek, Q.H. Qin, Application of hybrid-Trefftz element approach to transient heat conduction analysis, Computers & Structures, 58(1) (1996) 195-201.

[39] L. Cao, Q.H. Qin, N. Zhao, Application of DRM-Trefftz and DRM-MFS to Transient Heat Conduction Analysis, Recent Patents on Space Technology (Open access), 2 (2010) 41-50.

[40] N. Zhao, L.-L. Cao, Q.-H. Qin, Application of Trefftz Method to Heat Conduction Problem in Functionally Graded Materials, Recent Patents on Space Technology, 1(2) (2011) 158-166.

[41] Q.H. Qin, Hybrid Trefftz finite-element approach for plate bending on an elastic foundation, Applied Mathematical Modelling, 18(6) (1994) 334-339.

[42] Q.H. Qin, Postbuckling analysis of thin plates by a hybrid Trefftz finite element method, Computer Methods in Applied Mechanics and Engineering, 128(1) (1995) 123-136.

[43] Q.H. Qin, Transient plate bending analysis by hybrid Trefftz element approach, Communications in Numerical Methods in Engineering, 12(10) (1996) 609-616.

[44] Q.H. Qin, Postbuckling analysis of thin plates on an elastic foundation by HT FE approach, Applied Mathematical Modelling, 21(9) (1997) 547-556.

[45] F. Jin, Q.H. Qin, A variational principle and hybrid Trefftz finite element for the analysis of Reissner plates, Computers & Structures, 56(4) (1995) 697-701.

[46] J. Jirousek, A. Wroblewski, Q.H. Qin, X. He, A family of quadrilateral hybrid-Trefftz p-elements for thick plate analysis, Computer Methods in Applied Mechanics and Engineering, 127(1) (1995) 315-344.

[47] Q.H. Qin, Hybrid-Trefftz Finite-Element Method for Reissner Plates on an Elastic-Foundation, Computer Methods in Applied Mechanics and Engineering, 122(3-4) (1995) 379-392.

[48] Q.H. Qin, S. Diao, Nonlinear analysis of thick plates on an elastic foundation by HT FE with p-extension capabilities, International Journal of Solids and Structures, 33(30) (1996) 4583-4604.

[49] Q.H. Qin, Nonlinear analysis of thick plates by HT FE approach, Computers & Structures, 61(2) (1996) 271-281.

[50] C.-Y. Lee, Q.H. Qin, H. Wang, Trefftz functions and application to 3D elasticity, Computer Assisted Mechanics and Engineering Sciences, 15 (2008) 251-263.

[51] Q.H. Qin, Variational formulations for TFEM of piezoelectricity, International Journal of Solids and Structures, 40(23) (2003) 6335-6346.

[52] Q.H. Qin, Solving anti-plane problems of piezoelectric materials by the Trefftz finite element approach, Computational Mechanics, 31(6) (2003) 461-468.

[53] Q.H. Qin, Mode III fracture analysis of piezoelectric materials by Trefftz BEM, Structural Engineering and Mechanics, 20(2) (2005) 225-240.

[54] Q.H. Qin, Fracture Analysis of Piezoelectric Materials by Boundary and Trefftz Finite Element Methods, WCCM VI in conjunction with

APCOM'04, Sept. 5-10, 2004, Beijing, China, (2004).

[55] Q.H. Qin, Trefftz Plane Element of Piezoelectric Plate with p-Extension Capabilities, IUTAM Symposium on Mechanics and Reliability of Actuating Materials, (2006) 144-153.

[56] Q.H. Qin, K.-Y. Wang, Application of hybrid-Trefftz finite element method fractional contact problems, Computer Assisted Mechanics and Engineering Sciences, 15 (2008) 319-336.

[57] K. Wang, Q.H. Qin, Y. Kang, J. Wang, C. Qu, A direct constraint-Trefftz FEM for analysing elastic contact problems, International Journal for Numerical Methods in Engineering, 63(12) (2005) 1694-1718.

[58] M. Dhanasekar, K.Y. Wang, Q.H. Qin, Y.L. Kang, Contact analysis using Trefftz and interface finite elements, Computer Assisted Mechanics and Engineering Sciences, 13(3) (2006) 457-471.

[59] H. Wang, Q.H. Qin, Hybrid FEM with fundamental solutions as trial functions for heat conduction simulation, Acta Mechanica Solida Sinica, 22(5) (2009) 487-498.

[60] C. Cao, Q.-H. Qin, Hybrid fundamental solution based finite element method: theory and applications, Advances in Mathematical Physics, 2015 (2015).

[61] Q.H. Qin, Fundamental Solution Based Finite Element Method, J Appl Mech Eng, 2 (2013) e118.

[62] Z.-J. Fu, W. Chen, Q.-H. Qin, Hybrid Finite Element Method Based on Novel General Solutions for Helmholtz-Type Problems, Computers Materials and Continua, 21(3) (2011) 187.

[63] Y.-T. Gao, H. Wang, Q.-H. Qin, Orthotropic Seepage Analysis using Hybrid Finite Element Method, Journal of Advanced Mechanical Engineering, 2(1) (2015) 1-13.

[64] H. Wang, Q.H. Qin, Fundamental solution-based hybrid finite element analysis for non-linear minimal surface problems, Recent Developments in Boundary Element Methods: A Volume to Honour Professor John T. Katsikadelis, 309 (2010).

[65] H. Wang, Q.-H. Qin, Fundamental-solution-based hybrid FEM for plane elasticity with special elements, Computational Mechanics, 48(5) (2011) 515-528.

[66] H. Wang, Q.H. Qin, W. Yao, Improving accuracy of opening-mode stress intensity factor in two-dimensional media using fundamental solution based finite element model, Australian Journal of Mechanical Engineering, 10(1) (2012) 41-51.

[67] Q.H. Qin, H. Wang, Special Elements for Composites Containing Hexagonal and Circular Fibers, International Journal of Computational Methods, 12(04) (2015) 1540012.

[68] H. Wang, Q.H. Qin, Special fiber elements for thermal analysis of fiber-reinforced composites, Engineering Computations, 28(8) (2011) 1079-1097.

[69] H. Wang, Q.H. Qin, A fundamental solution based FE model for thermal analysis of nanocomposites, in: Boundary elements and other mesh Reduction methods XXXIII', 33rd International Conference on Boundary Elements and other Mesh Reduction Methods, ed. CA Brebbia and V. Popov, WIT Press, UK, 2011, pp. 191-202.

[70] H. Wang, Q.H. Qin, Implementation of fundamental-solution based hybrid finite element model for elastic circular inclusions, in: Proceedings of the Asia-Pacific Congress for Computational Mechanics, 11th-14th Dec, 2013.

[71] C. Cao, Q.H. Qin, A. Yu, Hybrid fundamental-solution-based FEM for piezoelectric materials, Computational Mechanics, 50(4) (2012) 397-412.

[72] C. Cao, A. Yu, Q.-H. Qin, A new hybrid finite element approach for plane piezoelectricity with defects, Acta Mechanica, 224(1) (2013) 41-61.

[73] H. Wang, Q.H. Qin, Fracture analysis in plane piezoelectric media using hybrid finite element model, in: International Conference of fracture, 2013.

[74] C. Cao, Q.-H. Qin, A. Yu, A new hybrid finite element approach for three-dimensional elastic problems, Archives of Mechanics, 64(3) (2012) 261-292.

[75] L.-L. Cao, Q.-H. Qin, N. Zhao, Hybrid graded element model for transient heat conduction in functionally graded materials, Acta Mechanica Sinica, 28(1) (2012) 128-139.

[76] L. Cao, H. Wang, Q.-H. Qin, Fundamental solution based graded element model for steady-state heat transfer in FGM, Acta Mechanica Solida Sinica, 25(4) (2012) 377-392.

[77] H. Wang, Q.-H. Qin, Boundary integral based graded element for elastic analysis of 2D functionally graded plates, European Journal of Mechanics-A/Solids, 33 (2012) 12-23.

[78] H. Wang, Q.H. Qin, FE approach with Green's function as internal trial function for simulating bioheat transfer in the human eye, Archives of Mechanics, 62(6) (2010) 493-510.

[79] H. Wang, Q.H. Qin, Computational bioheat modeling in human eye with local blood perfusion effect, Human Eye Imaging and Modeling, (2012).

[80] H. Wang, Q.-H. Qin, A fundamental solution-based finite element model for analyzing multi-layer skin burn injury, Journal of Mechanics in Medicine and Biology, 12(05) (2012) 1250027.

[81] Z.-W. Zhang, H. Wang, Q.-H. Qin, Transient bioheat simulation of the laser-tissue interaction in human skin using hybrid finite element formulation, Molecular & Cellular Biomechanics, 9(1) (2012) 31-53.

[82] Z.W. Zhang, H. Wang, Q.H. Qin, Analysis of transient bioheat transfer in the human eye using hybrid finite element model, in: Applied Mechanics and Materials, Trans Tech Publ, 2014, pp. 356-361.

[83] C. Cao, Q.-H. Qin, A. Yu, A novel boundary-integral based finite element method for 2D and 3D thermo-elasticity problems, Journal of Thermal Stresses, 35(10) (2012) 849-876.

[84] Q.H. Qin, H. Wang, Special circular hole elements for thermal analysis in cellular solids with multiple circular holes, International Journal of Computational Methods, 10(04) (2013) 1350008.

[85] H. Wang, Q.-H. Qin, A new special element for stress concentration analysis of a plate with elliptical holes, Acta Mechanica, 223(6) (2012) 1323-1340.

[86] Q.H. Qin, H. Wang, Fundamental solution based FEM for nonlinear thermal radiation problem, in: 12th International Conference on Boundary Element and Meshless Techniques (BeTeq 2011), ed. EL Albuquerque, MH Aliabadi, EC Ltd, Eastleigh, UK, 2011, pp. 113-118.

[87] C. Cao, A. Yu, Q.-H. Qin, A novel hybrid finite element model for modeling anisotropic composites, Finite Elements in Analysis and Design, 64 (2013) 36-47.

[88] C. Cao, A. Yu, Q.H. Qin, Mesh reduction strategy: Special element for modelling anisotropic materials with defects, Boundary Elements and Other Mesh Reduction Methods XXXVI, 56 (2013) 61.

[89] H. Wang, Q.-H. Qin, Fundamental-solution-based finite element model for plane orthotropic elastic bodies, European Journal of Mechanics-A/Solids, 29(5) (2010) 801-809.

[90] H. Wang, L. Cao, Q.-H. Qin, Hybrid Graded Element Model for Nonlinear Functionally Graded Materials, Mechanics of Advanced Materials and Structures, 19(8) (2012) 590-602.

[91] Q.H. Qin, Y. Huang, The study and prospect of solutions on the nonlinear problem of plates, Chinese Journal of Engineering Mechanics, 7(4) (1990) 10-18.

[92] S. Yu, X. Feng, Q.H. Qin, On several problems of macro/micro failure mechanics, Key Engineering Materials, 145 (1998) 51-60.

[93] Q.H. Qin, On nonlinear solutions for dislocations in a bimaterial solid, Structural integrity and fracture: proceedings of the International Conference on Structural Integrity and Fracture, SIF 2002, Perth, Australia, 25-27 September 2002, (2002) 71-79.

[94] X.-C. Hong, Q.H. Qin, Bifurcation of Limit Cycles in a Cubic Hamiltonian System with Some Special Perturbed Terms, International Journal of Mathematical Sciences, 6(2) (2007) 187-198.

[95] X.-C. Hong, Q.-H. Qin, Bifurcation of Limit Cycles in Two Given Planar Polynomial Systems, Recent Advances in Computer Science and Information Engineering, (2012) 705-713.

[96] L. Marin, D. Lesnic, The method of fundamental solutions for nonlinear functionally graded materials, International journal of solids and structures, 44(21) (2007) 6878-6890.

[97] L. Gray, T. Kaplan, J. Richardson, G.H. Paulino, Green's functions and boundary integral analysis for exponentially graded materials: heat conduction, Journal of Applied Mechanics, 70(4) (2003) 543-549.

[98] L. Gray, T. Kaplan, J. Richardson, G.H. Paulino, Green's functions and boundary integral analysis for exponentially graded materials: heat conduction, Journal of Applied Mechanics, 70(4) (2003) 543-549.

[99] C. Chao, M. Shen, On bonded circular inclusions in plane thermoelasticity, Journal of applied mechanics, 64(4) (1997) 1000-1004.