On Interval Valued Intuitionistic (S, T)-fuzzy Hv-ideals

Arvind Kumar Sinha¹, Manoj Kumar Dewangan²
Department of Mathematics, NIT Raipur, Chhattisgarh, India

ABSTRACT

Atanassov introduced the concept of the interval valued intuitionistic fuzzy sets. By using this we introduce the notion of interval valued intuitionistic -Hv, fuzzy -ideals of an Hv -ring with respect to a t-norm T and an s-norm S. Also some of their characteristic properties are described. The homomorphic image and the inverse image are investigated.

Keywords: Hv-ideal, interval valued intuitionistic (S, T)-fuzzy Hv-ideal, interval valued intuitionistic (S, T)-fuzzy ideal

I. INTRODUCTION

The concept of hyperstructure was introduced in 1934 by Marty [1]. Hyperstructures have many applications to several branches of pure and applied sciences. Vougiouklis [2] introduced the notion of Hv-structures, and Davvaz [3] surveyed the theory of Hv-structures.

After the introduction of fuzzy sets by Zadeh [4], there have been a number of generalizations of this fundamental concept. The notion of intuitionistic fuzzy sets introduced by Atanassov [5] is one among them. For more details on intuitionistic fuzzy sets, we refer the reader to [6, 7].

In [8] Biswas applied the concept of intuitoinistic fuzzy sets to the theory of groups and studied intuitionistic fuzzy subgroups of a group. In [9] Kim et al. introduced the notion of fuzzy subquasigroups of a quasigroup. In [10] Kim and Jun introduced the concept of fuzzy ideals of a semigroup. Zhan et al. [11] introduced the notion of an interval valued intuitionistic (S, T) -fuzzy Hv -submodule of an Hv -module. This paper continues this line of research for fuzzy Hv -ideal of Hv -ring. In this paper, we introduce the notion of interval valued intuitionistic (S, T) -fuzzy Hv -ideals of an Hv -ring and describe the characteristic properties. We give the homomorphic image and the inverse image.

The paper is organized as follows: in section 2 some fundamental definitions on Hv-structures and fuzzy sets are explored, in section 3 we define interval valued intuitionistic (S, T)-fuzzy Hv-ideals and establish some useful properties.

II. METHODS AND MATERIAL

1. Basic Definitions

We first give some basic definitions for proving the further results.

Definition 1.1 [12] Let X be a non-empty set. A mapping \( \mu : X \rightarrow [0,1] \) is called a fuzzy set in X. The complement of \( \mu \), denoted by \( \mu^c \), is the fuzzy set in X given by
\[
\mu^c(x) = 1 - \mu(x) \quad \forall x \in X.
\]

Definition 1.2 [12] Let \( f \) be a mapping from a set X to a set Y. Let \( \mu \) be a fuzzy set in X and \( \lambda \) be a fuzzy set in Y. Then the inverse image \( f^{-1}(\lambda) \) of \( \lambda \) is a fuzzy set in X defined by
\[
f^{-1}(\lambda)(x) = \lambda(f(x)) \quad \forall x \in X.
\]

The image \( f(\mu) \) of \( \mu \) is the fuzzy set in Y defined by
\[ f(\mu)(y) = \begin{cases} 
\sup_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \emptyset \\
0, & \text{otherwise}
\end{cases} \]

For all \( y \in Y \).

**Definition 1.3** [12] An intuitionistic fuzzy set \( A \) in a non-empty set \( X \) is an object having the form \( A = \{(x, \mu_a(x), \lambda_a(x)) : x \in X\} \), where the functions \( \mu_a : X \to [0,1] \) and \( \lambda_a : X \to [0,1] \) denote the degree of membership and degree of non-membership of each element \( x \in X \) to the set \( A \) respectively and \( 0 \leq \mu_a(x) + \lambda_a(x) \leq 1 \) for all \( x \in X \). We shall use the symbol \( A = \{\mu_a, \lambda_a\} \) for the intuitionistic fuzzy set \( A = \{(x, \mu_a(x), \lambda_a(x)) : x \in X\} \).

**Definition 1.4** [12] Let \( A = \{\mu_a, \lambda_a\} \) and \( B = \{\mu_b, \lambda_b\} \) be intuitionistic fuzzy sets in \( X \). Then
1. \( A \subseteq B \iff \mu_a(x) \leq \mu_b(x) \) and \( \lambda_a(x) \leq \lambda_b(x) \),
2. \( A^c = \{(x, \lambda_a(x), \mu_a(x)) : x \in X\} \),
3. \( A \cap B = \{(x, \min\{\mu_a(x), \mu_b(x)\}, \max\{\lambda_a(x), \lambda_b(x)\}) : x \in X\} \),
4. \( A \cup B = \{(x, \max\{\mu_a(x), \mu_b(x)\}, \min\{\lambda_a(x), \lambda_b(x)\}) : x \in X\} \),
5. \( \square A = \{(x, \mu_a(x), \mu_a(x)) : x \in X\} \),
6. \( \Diamond A = \{(x, \lambda_a(x), \lambda_a(x)) : x \in X\} \).

**Definition 1.5** [13] Let \( G \) be a non-empty set and \( \ast : G \times G \to \wp^*(G) \) be a hyperoperation, where \( \wp^*(G) \) is the set of all the non-empty subsets of \( G \). Where \( A \ast B = \bigcup_{a \in A, b \in B} a \ast b, \forall A, B \subseteq G \).

The \( \ast \) is called weak commutative if \( x \ast y \cap y \ast x \neq \emptyset, \forall x, y \in G \).

The \( \ast \) is called weak associative if \( (x \ast y) \ast z \cap x \ast (y \ast z) \neq \emptyset, \forall x, y, z \in G \).

A hyperstructure \( (G, \ast) \) is called an \( H_{\ast} \)-group if
1. \( \ast \) is weak associative,
2. \( a \ast G = G \ast a = G, \forall a \in G \) (Reproduction axiom).

**Definition 1.6** [14] Let \( G \) be a hypergroup (or \( H_{\ast} \)-group) and let \( \mu \) be a fuzzy subset of \( G \). Then \( \mu \) is said to be a fuzzy subhypergroup (or fuzzy \( H_{\ast} \)-subgroup) of \( G \) if the following axioms hold:
1. \( \min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in R \),
2. \( \forall x, y \in G \) such that \( x \in a \ast y \) and \( \min\{\mu(a), \mu(x)\} \leq \{\mu(y)\} \).

**Definition 1.7** [15] Let \( G \) be a hypergroup (or \( H_{\ast} \)-group). An intuitionistic fuzzy set \( A = \{\mu_a, \lambda_a\} \) of \( G \) is called intuitionistic fuzzy subhypergroup (or intuitionistic fuzzy \( H_{\ast} \)-subgroup) of \( G \) if the following axioms hold:
1. \( \min\{\mu_a(x), \mu_a(y)\} \leq \inf\{\mu_a(\alpha)\}, \forall x, y \in G \),
2. For all \( x, a \in G \) there exists \( y \in G \) such that \( x \in a \ast y \) and \( \min\{\mu(a), \mu(x)\} \leq \{\mu(y)\} \).

**Definition 1.8** [13] An \( H_{\ast} \)-ring is a system \( (R, +, \cdot) \) with two hyperoperations satisfying the ring-like axioms:
1. \((R, +)\) is an \( H_{\ast} \)-group, that is, \((x + y) + z = (x + (y + z)) \neq \emptyset \forall x, y \in R, a + R = R + a = R \forall a \in R;
2. \((R, \cdot)\) is an \( H_{\ast} \)-semigroup;
3. \((\cdot)\) is weak distributive with respect to \((+)\), that is, for all \( x, y, z \in R \),
\[(x \cdot (y + z)) \cap (x \cdot y + x \cdot z) \neq \emptyset, \]
\[(x + y) \cdot z \cap (x + y + z) \neq \emptyset.
\]

**Definition 1.9** [16] Let \( R \) be an \( H_{\ast} \)-ring. A nonempty subset \( I \) of \( R \) is called a left (resp., right) \( H_{\ast} \)-ideal if the following axioms hold:
1. \((I, +)\) is an \( H_{\ast} \)-subgroup of \((R, +)\),
2. \(R \cdot I \subseteq I \) (resp., \(I \cdot R \subseteq I \)).

**Definition 1.10** [16] Let \( (R, +, \cdot) \) be an \( H_{\ast} \)-ring and \( \mu \) a fuzzy subset of \( R \). Then \( \mu \) is said to be a left (resp., right) fuzzy \( H_{\ast} \)-ideal of \( R \) if the following axioms hold:
1. \( \min\{\mu(x), \mu(y)\} \leq \inf\{\mu(z) : z \in x + y\} \forall x, y \in R \),
(2) For all \( x, a \in R \) there exists \( y \in R \) such that
\[
x + a \leq y + a \quad \text{and} \quad \min\{\mu(a), \mu(x)\} \leq \mu(y),
\]
(3) For all \( x, a \in R \) there exists \( z \in R \) such that
\[
x + z \leq a \quad \text{and} \quad \min\{\mu(a), \mu(x)\} \leq \mu(z),
\]
(4) \( \mu(y) \leq \inf\{\mu(z) : z \in x \cdot y\} \) [respectively
\( \mu(x) \leq \inf\{\mu(z) : z \in x \cdot y\} \) \( \forall x, y \in R \). 

**Definition 1.11** [16] An intuitionistic fuzzy set \( A = \{\mu_a, \lambda_a\} \) in \( R \) is called a left (resp., right)
intuitionistic fuzzy \( H_t \)-ideal of \( R \) if
(1) \( \min\{\mu_a(x), \mu_a(y)\} \leq \inf\{\mu_a(z) : z \in x + y\} \)
\[
\max\{\lambda_a(x), \lambda_a(y)\} \geq \sup\{\lambda_a(z) : z \in x + y\}
\]
\( \forall x, y \in R; \)
(2) For all \( x, a \in R \) there exists \( y \in R \) such that
\[
x + a \leq y \quad \text{and} \quad \min\{\mu_a(a), \mu_a(x)\} \leq \mu_a(y)
\]
and
\[
\max\{\lambda_a(a), \lambda_a(x)\} \geq \lambda_a(y);
\]
(3) For all \( x, a \in R \) there exists \( z \in R \) such that
\[
x + z \leq a \quad \text{and} \quad \min\{\mu_a(a), \mu_a(x)\} \leq \mu_a(z)
\]
and
\[
\max\{\lambda_a(a), \lambda_a(x)\} \geq \lambda_a(z);
\]
(4) \( \mu_a(y) \leq \inf\{\mu_a(z) : z \in x \cdot y\} \) [respectively
\( \mu_a(x) \leq \inf\{\mu_a(z) : z \in x \cdot y\} \) \( \forall x, y \in R \) and
\[
\lambda_a(y) \geq \sup\{\lambda_a(z) : z \in x \cdot y\}
\]
\( \lambda_a(x) \geq \sup\{\lambda_a(z) : z \in x \cdot y\} \) \( \forall x, y \in R \).

**Definition 1.12** [17] By a \( t \)-norm \( T \), we mean a function
\( T : [0,1] \times [0,1] \rightarrow [0,1] \) satisfying the following conditions:
(i) \( T(x,1) = x \),
(ii) \( T(x,y) \leq T(x,z) \) if \( y \leq z \),
(iii) \( T(x,y) = T(y,x) \),
(iv) \( T(T(x,y),z) = T(x,T(y,z)) \).
For all \( x, y, z \in [0,1] \).

**Definition 1.13** [17] By a \( s \)-norm \( S \), we mean a function
\( S : [0,1] \times [0,1] \rightarrow [0,1] \) satisfying the following conditions:
(i) \( S(x,0) = x \),
(ii) \( S(x,y) \leq S(x,z) \) if \( y \leq z \),
(iii) \( S(x,y) = S(y,x) \),
(iv) \( S(x,S(y,z)) = S(S(x,y),z) \).
For all \( x, y, z \in [0,1] \).
It is clear that
\( T(\alpha, \beta) \leq \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq S(\alpha, \beta) \). For all
\( \alpha, \beta \in [0,1] \). 
By an interval number \( \tilde{a} \) we mean an interval \( [a^-, a^+] \)
where \( 0 \leq a^- \leq a^+ \leq 1 \). The set of all interval numbers is denoted by \( D[0,1] \). We also identify the interval
\( [a, a] \) by the number \( a \in [0,1] \).
For the interval numbers \( \tilde{a}_i = [a^-_i, a^+_i] \in D[0,1] \), \( i \in I \),
we define
\[
\max\{\tilde{a}_i, \tilde{b}_i\} = \left[\max\{a^-_i, b^-_i\}, \max\{a^+_i, b^+_i\}\right],
\]
\[
\min\{\tilde{a}_i, \tilde{b}_i\} = \left[\min\{a^-_i, b^-_i\}, \min\{a^+_i, b^+_i\}\right],
\]
\[
\inf\tilde{a}_i = \left[\bigwedge_{i \in I} a^-_i, \bigwedge_{i \in I} a^+_i\right],
\]
\[
\sup\tilde{a}_i = \left[\bigvee_{i \in I} a^-_i, \bigvee_{i \in I} a^+_i\right],
\]
and put
\( (1) \tilde{a}_1 \leq \tilde{a}_2 \iff a^-_1 \leq a^-_2 \) and \( a^+_1 \leq a^+_2 \),
\( (2) \tilde{a}_1 = \tilde{a}_2 \iff a^-_1 = a^-_1 \) and \( a^+_1 = a^+_2 \),
\( (3) \tilde{a}_1 < \tilde{a}_2 \iff \tilde{a}_1 \leq \tilde{a}_2 \) and \( \tilde{a}_1 \neq \tilde{a}_2 \),
\( (4) k\tilde{a} = \left[ka^-, ka^+\right], \) whenever \( 0 \leq k \leq 1 \).
It is clear that \( (D[0,1],\leq, \lor, \land) \) is a complete lattice
with \( 0 = [0,0] \) as least element and \( 1 = [1,1] \) as greatest
element.
By an interval valued fuzzy set \( F \) on \( X \) we mean the set
\( F = \left\{\left(x, [\mu^-_F(x), \mu^+_F(x)]\right) : x \in X\right\} \). Where \( \mu^-_F \) and
\( \mu^+_F \) are fuzzy subsets of \( X \) such that
\( \mu^-_F(x) \leq \mu^+_F(x) \) for all \( x \in X \). Put \( \tilde{\mu}_F(x) = \left[\mu^-_F(x), \mu^+_F(x)\right]. \) Then
\( F = \left\{\left(x, \tilde{\mu}_F(x)\right) : x \in X\right\}. \) where \( \tilde{\mu}_F : X \rightarrow D[0,1] \). 
If \( A, B \) are two interval valued fuzzy subsets of \( X \), then
we define
\( A \subseteq B \) if and only if for all \( x \in X \), \( \mu^-_A(x) \leq \mu^-_B(x) \) and \( \mu^+_A(x) \leq \mu^+_B(x) \),
\( A = B \) if and only if for all \( x \in X \), \( \mu^-_A(x) = \mu^-_B(x) \) and \( \mu^+_A(x) = \mu^+_B(x) \).
Also, the union, intersection and complement are defined as follows: let A; B be two interval valued fuzzy subsets of X, then

\[ A \cup B = \left\{ x \in X : \max \{ \mu_A(x), \mu_B(x) \}, \max \{ \mu_A(x), \mu_B(x) \} : x \in X \right\}, \]

\[ A \cap B = \left\{ x \in X : \min \{ \mu_A(x), \mu_B(x) \}, \min \{ \mu_A(x), \mu_B(x) \} : x \in X \right\}, \]

\[ A' = \left\{ x \in X : 1 - \mu_A(x), 1 - \mu_A(x) : x \in X \right\}. \]

According to Atanassov an interval valued intuitionistic fuzzy set on X is defined as an object of the form

\[ A = \left\{ x, \tilde{\mu}_A(x), \tilde{\lambda}_A(x) : x \in X \right\}, \]

where \( \tilde{\mu}_A(x) \) and \( \tilde{\lambda}_A(x) \) are interval valued fuzzy sets on X such that

\[ 0 \leq \sup \tilde{\mu}_A(x) + \sup \tilde{\lambda}_A(x) \leq 1 \text{ for all } x \in X. \]

For the sake of simplicity, in the following such interval valued intuitionistic fuzzy sets will be denoted by \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \).

2. Interval Valued Intuitionistic \((S,T)\)-Fuzzy \(H_v\)-Ideals

In what follows, let R denote an \( H_v \)-ring unless otherwise specified.

Definition 2.1 An interval valued intuitionistic fuzzy set \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) of R is called an interval valued intuitionistic \((S,T)\)-fuzzy \(H_v\)-ideal of R if the following conditions hold:

1. \( T(\tilde{\mu}_A(x), \tilde{\mu}_A(y)) \leq \inf_{\alpha \in [s,t]} \tilde{\mu}_A(\alpha) \) and
2. \( S(\tilde{\lambda}_A(x), \tilde{\lambda}_A(y)) \geq \sup_{\alpha \in [s,t]} \tilde{\lambda}_A(\alpha), \forall x, y \in R, \)
3. \( \forall x, a \in R \text{ there exists } y \in R \text{ such that } x \in a + y \text{ and } T(\tilde{\mu}_A(x), \tilde{\mu}_A(a)) \leq \tilde{\mu}_A(y) \) and
4. \( S(\tilde{\lambda}_A(x), \tilde{\lambda}_A(a)) \geq \tilde{\lambda}_A(y), \)
5. \( \forall x, a \in R \text{ there exists } z \in R \text{ such that } x \in z + a \text{ and } T(\tilde{\mu}_A(x), \tilde{\mu}_A(a)) \leq \tilde{\mu}_A(z) \) and
6. \( S(\tilde{\lambda}_A(x), \tilde{\lambda}_A(a)) \geq \tilde{\lambda}_A(z), \)

With any interval valued intuitionistic fuzzy set \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) of R there are connected two levels:

\[ U(\tilde{\mu}_A;[t,s]) = \{ x \in R : \tilde{\mu}_A(x) \geq [t,s] \}, \]

\[ L(\tilde{\lambda}_A;[t,s]) = \{ x \in R : \tilde{\lambda}_A(x) \leq [t,s] \}. \]

Theorem 2.2 Let T (resp. S) be an idempotent interval t-norm (resp. s-norm). Then \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) is an interval valued intuitionistic \((S,T)\)-fuzzy \(H_v\)-ideal of R and only if for all \( t, s \in [0,1], t \leq s, U(\tilde{\mu}_A;[t,s]) \) and \( L(\tilde{\lambda}_A;[t,s]) \) are \( H_v\)-ideals of R.

Proof Let \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) be an interval valued intuitionistic \((S,T)\)-fuzzy \(H_v\)-ideal of R. Then for every \( x, y \in U(\tilde{\mu}_A;[t,s]) \) we have \( \tilde{\mu}_A(x) \geq [t,s] \) and \( \tilde{\mu}_A(y) \geq [t,s] \). Hence

\[ T(\tilde{\mu}_A(x), \tilde{\mu}_A(y)) \geq T([t,s],[t,s]) = [t,s], \]

so \( \inf_{\alpha \in [s,t]} \tilde{\mu}_A(\alpha) \geq [t,s] \). Therefore \( \alpha \in U(\tilde{\mu}_A;[t,s]) \) for every \( \alpha \in x + y \), so \( x + y \subseteq U(\tilde{\mu}_A;[t,s]) \). Thus, for every \( a \in U(\tilde{\mu}_A;[t,s]) \), we have

\[ a + U(\tilde{\mu}_A;[t,s]) \subseteq U(\tilde{\mu}_A;[t,s]). \]

On the other hand, for \( x, a \in U(\tilde{\mu}_A;[t,s]) \) there exists \( y \in R \) such that \( x \in a + y \) and \( T(\tilde{\mu}_A(x), \tilde{\mu}_A(a)) \leq \tilde{\mu}_A(y) \) and \( T(\tilde{\mu}_A(x), \tilde{\mu}_A(a)) \geq \tilde{\mu}_A(y) \). But \( T(\tilde{\mu}_A(x), \tilde{\mu}_A(a)) \leq T([t,s],[t,s]) = [t,s] \), so \( \tilde{\mu}_A(y) \geq [t,s] \) that is, \( y \in U(\tilde{\mu}_A;[t,s]) \), whence \( U(\tilde{\mu}_A;[t,s]) \subseteq a + U(\tilde{\mu}_A;[t,s]) \), and, in consequence, \( U(\tilde{\mu}_A;[t,s]) = a + U(\tilde{\mu}_A;[t,s]) \). Similarly, we can prove that \( a + U(\tilde{\mu}_A;[t,s]) = U(\tilde{\mu}_A;[t,s]) + a \). That is, \( a + U(\tilde{\mu}_A;[t,s]) = U(\tilde{\mu}_A;[t,s]) = U(\tilde{\mu}_A;[t,s]) + a \).

This proves that \( U(\tilde{\mu}_A;[t,s]) \) is an \( H_v\)-subgroup of \((R,+)\).

If \( r \in R \) and \( x \in U(\tilde{\mu}_A;[t,s]) \) then \( \tilde{\mu}_A(x) \geq [t,s] \), which means that \( \inf_{\alpha \in [s,t]} \tilde{\mu}_A(\alpha) \geq [t,s] \).

So, \( \alpha \in U(\tilde{\mu}_A;[t,s]) \) for every \( \alpha \in r \cdot x \). Therefore,
\[ r \cdot x \subseteq U(\tilde{\mu}_A;[t,s]), \text{ i.e.} \]
\[ r \cdot U(\tilde{\mu}_A;[t,s]) \subseteq U(\tilde{\mu}_A;[t,s]). \]
This proves that \( U(\tilde{\mu}_A;[t,s]) \) is an \( H_v \)-ideal of \( R \). Similarly, we can show that \( L(\tilde{\lambda}_A;[t,s]) \) is an \( H_v \)-ideal of \( R \).

Conversely, assume that for every \([t,s] \in D[0,1]\) any non-empty \( U(\tilde{\mu}_A;[t,s]) \) is an \( H_v \)-ideal of \( R \). If \([t_0, s_0] = T(\tilde{\mu}_A(x), \tilde{\mu}_A(y)) \) for some \( x, y \in R \), then \( x, y \in U(\tilde{\mu}_A;[t_0, s_0]) \), and so \( x + y \subseteq U(\tilde{\mu}_A;[t_0, s_0]) \).

Therefore \( \inf_{\alpha \in U(\tilde{\mu}_A;[t_0, s_0])} \tilde{\mu}_A(\alpha) \geq T(\tilde{\mu}_A(x), \tilde{\mu}_A(y)) \). Now, if \([t_1, s_1] = T(\tilde{\mu}_A(a), \tilde{\mu}_A(x)) \) for some \( a, x \in R \), then \( a + x \in U(\tilde{\mu}_A;[t_1, s_1]) \), so there exists \( y \in U(\tilde{\mu}_A;[t_1, s_1]) \) such that \( x \in a + y \). But for \( y \in U(\tilde{\mu}_A;[t_1, s_1]) \) we have \( \tilde{\mu}_A(y) \geq [t_1, s_1] \), whence \( \tilde{\mu}_A(y) \geq T(\tilde{\mu}_A(a), \tilde{\mu}_A(x)) \).

Similarly, we can show that for \( a, x \in R \), there exists \( z \in R \) such that \( x \in z + a \) and \( \tilde{\mu}_A(z) \geq T(\tilde{\mu}_A(a), \tilde{\mu}_A(x)) \). If \([t_2, s_2] = \tilde{\mu}_A(x) \) for some \( x \in R \), then \( x \in U(\tilde{\mu}_A;[t_2, s_2]) \), and so \( r \cdot x \subseteq U(\tilde{\mu}_A;[t_2, s_2]) \) for every \( r \in R \). Therefore for every \( \alpha \in r \cdot x \), we have \( \alpha \in U(\tilde{\mu}_A;[t_2, s_2]) \), consequently \( \inf_{\alpha \in r \cdot x} \tilde{\mu}_A(\alpha) \geq [t_2, s_2] = \tilde{\mu}_A(x) \).

This proves that \( \tilde{\mu}_A \) is an interval valued \( T \)-fuzzy \( H_v \)-ideal of \( R \).

Similarly, we can show that \( \tilde{\lambda}_A \) is an interval valued \( S \)-fuzzy \( H_v \)-ideal of \( R \). Therefore, \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) is an interval valued intuitionistic \( (S,T) \)-fuzzy \( H_v \)-ideal of \( R \).

**Definition 2.3** Let \( f : X \rightarrow Y \) be a mapping and \( A = (\tilde{\mu}_A, \tilde{\lambda}_A), B = (\tilde{\mu}_B, \tilde{\lambda}_B) \) an interval valued intuitionistic sets \( X \) and \( Y \), respectively. Then the image \( f[A] = (f(\tilde{\mu}_A), f(\tilde{\lambda}_A)) \) of \( A \) is the interval valued intuitionistic fuzzy set of \( Y \) defined by

\[ f(\tilde{\mu}_A)(y) = \left\{ \begin{array}{ll}
sup_{z \in f^{-1}(y)} \tilde{\mu}_A(z), & f^{-1}(y) \neq \phi \\
[0,0], & f^{-1}(y) = \phi
\end{array} \right. \]

and

\[ f(\tilde{\lambda}_A)(y) = \left\{ \begin{array}{ll}
inf_{z \in f^{-1}(y)} \tilde{\lambda}_A(z), & f^{-1}(y) \neq \phi \\
[1,1], & f^{-1}(y) = \phi
\end{array} \right. \]

for all \( y \in Y \).

The inverse image \( f^{-1}(B) \) of \( B \) is an interval valued intuitionistic fuzzy set defined by \( f^{-1}(\tilde{\mu}_B)(x) = \tilde{\mu}_{f^{-1}(b)}(x) = \tilde{\mu}_B(f(x)), \)

\( f^{-1}(\tilde{\lambda}_B)(x) = \tilde{\lambda}_{f^{-1}(b)}(x) = \tilde{\lambda}_B(f(x)) \) for all \( x \in X \).

**Definition 2.4** [18] Let \( R \) and \( S \) be two \( H_v \)-rings. A mapping \( f : R \rightarrow S \) is called an \( H_v \)-homomorphism or weak homomorphism if for all \( x, y, r \in R \) the following relations hold: \( f(x + y) \cap (f(x) + f(y)) \neq \phi \) and \( f(r \cdot x) \cap r \cdot f(x) \neq \phi \).

\( f \) is called an inclusion homomorphism if \( f(x + y) \subseteq f(x) + f(y) \) and \( f(r \cdot x) \subseteq r \cdot f(x) \) for all \( x, y, r \in R \). Finally, \( f \) is called a strong homomorphism if for all \( x, y, r \in R \) we have \( f(x + y) = f(x) + f(y) \) and \( f(r \cdot x) = r \cdot f(x) \).

**Lemma 2.5** [18] Let \( R_1 \) and \( R_2 \) be two \( H_v \)-rings and \( f : R_1 \rightarrow R_2 \) a strong epimorphism. If \( S \) is an \( H_v \)-ideal of \( R_2 \), then \( f^{-1}(S) \) is an \( H_v \)-ideal of \( R_1 \).

**Theorem 2.6** Let \( R_1 \) and \( R_2 \) be two \( H_v \)-rings, \( f \) a strong epimorphism from \( R_1 \) into \( R_2 \) and \( T \) (resp. \( S \)) an idempotent interval t-norm (resp. s-norm).

(i) If \( A = (\tilde{\mu}_A, \tilde{\lambda}_A) \) is an interval valued intuitionistic \( (S,T) \)-fuzzy \( H_v \)-ideal of \( R_1 \), then the image \( f[A] \) of \( A \) is an interval valued intuitionistic \( (S,T) \)-fuzzy \( H_v \)-ideal of \( R_2 \).

(ii) If \( B = (\tilde{\mu}_B, \tilde{\lambda}_B) \) is an interval valued intuitionistic \( (S,T) \)-fuzzy \( H_v \)-ideal of \( R_2 \), then the inverse image...
$f^{-1}(B)$ of $B$ is an interval valued intuitionistic $(S,T)$-fuzzy $H_v$-ideal of $R_i$.

**Proof** (i) Let $A = (\bar{\mu}_A, \bar{\lambda}_A)$ be an interval valued intuitionistic $(S,T)$-fuzzy $H_v$-ideal of $R_i$. By

Theorem 2.2, $U(\bar{\mu}_A;[t,s])$ and $L(\bar{\lambda}_A;[t,s])$ are $H_v$-ideals of $R_i$ for every $[t,s] \in D[0,1]$. Therefore, by Lemma 3.5, $f(U(\bar{\mu}_A;[t,s]))$ and $f(L(\bar{\lambda}_A;[t,s]))$ are $H_v$-ideals of $R_2$. But

$U(f(\bar{\mu}_A);[t,s]) = f(U(\bar{\mu}_A;[t,s]))$ and

$L(f(\bar{\lambda}_A);[t,s]) = f(L(\bar{\lambda}_A;[t,s]))$, so

$U(f(\bar{\mu}_A);[t,s])$ and $L(f(\bar{\lambda}_A);[t,s])$ are $H_v$-ideals of $R_2$. Therefore $f[A]$ is an interval valued intuitionistic $(S,T)$-fuzzy $H_v$-ideal of $R_2$.

(ii) For any $x, y \in R$ and $\alpha \in x + y$, we have

$\beta_{f^{-1}(B)}(\alpha) = \bar{\mu}_B(f(\alpha)) \geq T(\bar{\mu}_B(f(x)), \bar{\mu}_B(f(y))) = T(\bar{\mu}_{f^{-1}(B)}(x), \bar{\mu}_{f^{-1}(B)}(y))$.

Therefore

$\inf_{\alpha x y} \bar{\mu}_{f^{-1}(B)}(\alpha) \geq T(\bar{\mu}_{f^{-1}(B)}(x), \bar{\mu}_{f^{-1}(B)}(y))$. For

$x, a \in R_2$, there exists $y \in R_2$ such that $x = a + y$.

Thus $f(x) \in f(a) + f(y)$ and

$T(\bar{\mu}_{f^{-1}(B)}(x), \bar{\mu}_{f^{-1}(B)}(y)) = T(\bar{\mu}_B(f(x)), \bar{\mu}_B(f(y))) \leq \bar{\mu}_B(f(x)) = \bar{\mu}_{f^{-1}(B)}(y)$.

In the same manner, we can show that for $x, a \in R_2$

there exists $z \in R_2$ such that $x = z + a$ and

$T(\bar{\mu}_{f^{-1}(B)}(x), \bar{\mu}_{f^{-1}(B)}(a)) \leq \bar{\mu}_{f^{-1}(B)}(z)$.

It is not difficult to see that, for all $x \in R_2, r \in R$ and $\alpha \in r \cdot x$, we have

$\bar{\mu}_{f^{-1}(B)}(\alpha) = \bar{\mu}_B(f(\alpha)) \geq \bar{\mu}_B(f(x)) = \bar{\mu}_{f^{-1}(B)}(x)$,

whence $\inf_{\alpha \in f^{-1}(B)} \bar{\mu}_{f^{-1}(B)}(\alpha) \geq \bar{\mu}_{f^{-1}(B)}(x)$.

This completes the proof that $\bar{\mu}_{f^{-1}(B)}$ is an interval valued $T$-ideal of $R_i$.

Similarly, we can prove $\bar{\lambda}_{f^{-1}(B)}$ is an interval valued $S$-fuzzy $H_v$-ideal of $R_i$. Therefore $f^{-1}(B)$ is an interval valued intuitionistic $(S,T)$-fuzzy $H_v$-ideal of $R_i$.

### III. REFERENCES


