Application of Parameter Expansion Method for Nonlinear Singularly Perturbed Boundary Value Problems (BVP'S)
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ABSTRACT
In this paper, we present a parameter expansion method for two-point nonlinear singularly perturbed boundary value problem for second order ordinary differential equation. Newton linearisation scheme is used to linearise the nonlinear problem.

Keywords: Boundary Value Problems, Newton Linearisation Scheme

I. INTRODUCTION
Consider a nonlinear singularly perturbed second order boundary value problem \( G_{\varepsilon y} \) defined by

\[
G_{\varepsilon y} = \varepsilon y''(x) - f(x, y, y', \varepsilon) y'(x) = 0, \quad a \leq x \leq b
\]

\( y(a, \varepsilon) = \alpha \) \hspace{2cm} \( y(b, \varepsilon) = \beta \)

for small positive values of the parameter \( \varepsilon \), satisfying \( 0 \leq \varepsilon \leq \varepsilon_0 \) for some \( \varepsilon_0 \), while \( a, b, \phi, \beta \) are independent of \( \varepsilon_0 \).

The Newton’s scheme for Taylor’s series expansion given by

\[
\frac{\partial G}{\partial y} = -f_y y'; \quad \frac{\partial G}{\partial y'} = - (f_y y' - f); \quad \frac{\partial G}{\partial y''} = \varepsilon
\]

(5)

Substituting (5) into (4), we have

\[
G_k + \Delta y_k \left[ -y_k \frac{\partial f_{yk}}{\partial y_k} \right] + \Delta y'_k \left[ -(f_k - y_k' \frac{\partial f_{yk}}{\partial y'_k}) \right] + \Delta y''_k (\varepsilon) = 0
\]

(6)

where \( \Delta y^{(j)}_k (x) = y^{(j)}_{k+1} (x) - y^{(j)}_k (x) \)

The Newton’s linearisation leads to the use of the following iteration;

\[
\varepsilon y_{k+1}^{x+1}(x) - \left( f_k + y_k'(x) \frac{\partial f(x)}{\partial y'_k} \right) y_{k+1}^{x+1}(x) - \left( y_k' \frac{\partial f_{yk}}{\partial y'_k} \right) y_{k+1}^{x+1}(x) = 0
\]

\[
\varepsilon y_k'(x) - y_k(x) y'_k(x) \frac{\partial f_{yk}}{\partial y'_k} - y_k'(x) \frac{\partial f}{\partial y_k} - G_k
\]

(7)

\[
y_{k+1}(a, \varepsilon) = \phi
\]

(8)

\[
y_{k+1}(b, \varepsilon) = \beta
\]

(9)

The Method of Parameter Expansion

The method seeks an asymptotic expansion based on the idea of Okoro et al [1], which converts the
singly perturbed problem to a system of ordinary differential equations for which the solutions are relatively easier to obtain. The ordinary differential equations are reduced to algebraic equations using the perturbed collocation method described in [1]. In order to solve (7) with this method, we seek a patched solution in two regions, namely the boundary layer region $I_{BL}$ and outside the boundary layer region $I_{OBL}$ where

$I = \{x : G \leq x \leq b\} = I_{OBL} \cup I_{BL}$

Without any loss of generality, we set

$I_{OBL} = \{x : a_i \leq x \leq b_i\}$
$I_{BL} = \{x : a_2 \leq x \leq b_2\}$

Where $a < a_1 < b_1 = a_2 < b_2 = b$

In the region $I_{OBL}$, we seek a smooth collocation solution of the form $y_{1,N}(x)$ and in the region $I_{BL}$, we seek the parameter expansion $y_{2,M}(x, \varepsilon)$. In the smooth solution, let,

$y_{1,N}(x) = \sum_{i=1}^{N} a_i x^{i-1}, \ a_i \leq x \leq b_1$  \hspace{1cm} (10)

satisfies exactly the slightly perturbed collocation equations

$$
eq y_{x,i}^{*} - \left(f_i + y'_{x,i} + y'_{x,i}(x) \frac{\partial f}{\partial y} \right) y_{x,i} = \left(y_{x,i} + \frac{\partial f}{\partial y} \right) y_{x,i} - y_{x,i} y_{x,i} \left(f_i + y'_{x,i} \frac{\partial f}{\partial y} \right) + H_N(x)$$.  

Where

$x_i = a_i + \frac{(b_i - a_i)^j}{N+1}; \ i = 1, 2, \ldots, N$  \hspace{1cm} (12)

And

$T_N^*(x_j) = T_N(2x_j - 1); \ a \leq x_j \leq b$  \hspace{1cm} (14)

is the shifted Chebyshev polynomial, and

$t_{1,k} (k = 1, 2)$ are arbitrary constants to be determined

and

$T_N(x) = \cos[N \arccos(x)]; N \geq 0, \ [-1, 1]$

The zeros of $T_N^*(x)$ are given by

$x_j = \frac{1}{2} \left((a + b) - (a - b) \cos \left((2j - 1) \frac{\pi}{2N}\right)\right); j = 1, \ldots, N$

Also, $y_{1,N}(x)$ must satisfy the arbitrary conditions

$y_{1,N}(a, \varepsilon) = \phi$  \hspace{1cm} (15)

and

$y_{1,N}(b, \varepsilon) = y_{2,M}(a_2, \varepsilon)$  \hspace{1cm} (16)

The Chebyshev perturbation (14) is well-known to yield an accurate approximation.

On substituting (10) in (11) we obtain N collocation equations. Two extra equations are obtain using (15) and (16).

Altogether, we have (N + 2) collocation equations which give the unique values of $a_0, a_1, \ldots, a_N, \mu_1, \text{and } \mu_{1,2}$. Also, inside the boundary layer region $I_{BL}$, we seek a uniform valid parameter expansion in the form.

$y_{2,N}(x, \varepsilon) = \sum_{j=1}^{N} g_j(x) \varepsilon^{j-1}; \ a_2 \leq x \leq b_2$  \hspace{1cm} (17)

which satisfies the following perturbed two-dimensional form of equation (7)
\[
\frac{\partial^2 y_{2,M}(x, \varepsilon)}{\partial x^2} + p(x) \frac{\partial y_{2,M}(x, \varepsilon)}{\partial x} + q(x)y_{2,M}(x, \varepsilon) + G_i(x, \varepsilon) = H_{2,M}(x, \varepsilon)
\]  
(18)

where

\[
p(x) = - \left( f_k(y'_k(x)) \frac{\partial f}{\partial y_k} \right); \quad q(x) = - \left( y'_k(x) \frac{\partial f_{jk}}{\partial y_k} \right)
\]

\[
G_i(x, \varepsilon) = y''_k(x) + y_k(x)y'_k(x) \frac{\partial f_{jk}}{\partial y_k} - y'_k(x) \left( f_k + y'_k(x) \frac{\partial f}{\partial y_k} \right) - G_k
\]

\[
H_{2,M}(x, \varepsilon) = \sum_{k=0}^{2} \mu_{2,k} T_{M-1}^*(x); \quad \alpha_2 \leq x \leq \beta_2
\]

Also, \( y_{2,M}(x, \varepsilon) \) must satisfy the following conditions

\[
\begin{align*}
y_{2,M}(a_2, \varepsilon) &= y_{1,N}(b_1) \\
y_{2,M}(b_2, \varepsilon) &= \beta
\end{align*}
\]  
(19)

and

\[
\begin{align*}
y_{2,M}(x, 1) &= \phi(x) \\
y_{2,M}(x, 0) &= \varepsilon(x)
\end{align*}
\]  
(20)

where \( \phi(x) \) and \( \varepsilon(x) \) are obtained from (1) when \( \varepsilon = 1 \) and when \( \varepsilon = 0 \) respectively.

Collocating equation (19) at points \( \varepsilon_i \), we obtain

\[
\varepsilon_i \frac{\partial^2 y_{2,M}(x, \varepsilon_i)}{\partial x^2} + p(x) \frac{\partial y_{2,M}(x, \varepsilon_i)}{\partial x} + q(x)y_{2,M}(x, \varepsilon_i) + G_i(x, \varepsilon_i) = H_{2,M}(x, \varepsilon_i)
\]  
(21)

where

\[
\varepsilon_i = - \frac{i}{M + 2}; \quad I = 1, 2, \ldots, M+1
\]  
(22)

Thus, we obtain \((M + 1)\) second order ordinary differential equations in \((M + 3)\) unknown functions, \( g_1(x) \), \( g_2(x) \), \( g_3(x) \), \ldots, \( g_M(x) \), \( \mu_{2,1} \) and \( \mu_{2,2} \).

The arbitrary \( \mu \)-functions are then eliminated to give a set of \((M - 2)\) second order ordinary differential equations. Two extra equations are obtained using (19). Altogether, we have \(M\) second order ordinary differential equation. The \(M\) second order ordinary differential equations are then perturbed and collocated in the same manner as in (11). Equations (20) are satisfied at the Chebyshev points \( x_i (i = 1, 2, 3, \ldots, M) \). These equations together with (12), (15) and (16) give the values of the constants \( a_i, \ v_{ij} \ (i = 1, 2, \ldots, N; j = 1, 2, \ldots, M) \) for the required approximation.
\[ y_{1,N}(x) = \sum_{j=1}^{N} a_j x^{j-1}; \quad a_1 \leq x \leq b_1 \]  

(23)

\[ y(x) \rightleftharpoons y_{N}(x) = y_{2,M}(x, \varepsilon) = \sum_{j=1}^{M} g_{i,j} x^{i-1} \varepsilon^{j-1}; \quad a_2 \leq x \leq b_2 \]

A Worked Example consider the nonlinear second order bvp

\[ \varepsilon y''(x) - y(x)y'(x) = 0, \quad -1 \leq x \leq 1 \]

With the boundary conditions

\[ y(-1) = -\tanh \left( \frac{4}{\varepsilon} \right) \quad \text{and} \quad y(1) = \tanh \left( \frac{4}{\varepsilon} \right) \]

The analytical solution is given by

\[ y(x) = \tanh \left( \frac{4x}{\varepsilon} \right) \]

The Newton’s iterates using (7) on (5) are given by

\[ \varepsilon y''_{N,K+1}(x, \varepsilon) - y_{N,K}(x, \varepsilon)y'_{N,K+1}(x, \varepsilon) - y_{N,K}(x, \varepsilon)y_{N,K}(x, \varepsilon) = -y_{N,K}(x, \varepsilon)y'_{N,K}(x, \varepsilon); \quad x \in [-1,1] \]

and

\[ y_{N,K+1}(-1, \varepsilon) = -\tanh \left( \frac{4}{\varepsilon} \right) \]

\[ y_{N,K+1}(1, \varepsilon) = \tanh \left( \frac{4}{\varepsilon} \right) \]

For K = 0, the initial approximation used is

\[ y_{N,K}(x) = \tanh \left( \frac{4x}{\varepsilon} \right) \]

\[ \begin{array}{|c|c|c|}
\hline
\varepsilon & \text{Standard collocation method} & \text{Parameter Expansion method} \\
\hline
10^{-2} & 1.516 \times 10^1 & 1.516 \times 10^1 \\
10^{-3} & 1.824 \times 10^{-2} & 2.053 \times 10^{-2} \\
10^{-4} & 1.939 \times 10^{-3} & 5.281 \times 10^{-3} \\
10^{-5} & 2.226 \times 10^{-3} & 2.172 \times 10^{-3} \\
10^{-6} & 2.251 \times 10^{-4} & 7.399 \times 10^{-4} \\
10^{-7} & 2.283 \times 10^{-4} & 4.052 \times 10^{-4} \\
10^{-8} & 2.487 \times 10^{-5} & 2.459 \times 10^{-5} \\
10^{-9} & 2.842 \times 10^{-5} & 2.725 \times 10^{-5} \\
\hline
\end{array} \]

Table 1: Error Estimates for Case N = 5, M = 4
II. CONCLUSION

The numerical results show that the accuracy of the proposed method when compared with the standard collocation tan method improves as $\varepsilon$ tends to zero.

III. REFERENCES

