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Application of Parameter Expansion Method for Nonlinear Singularly Perturbed Boundary Value Problems (BVP'S)

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ABSTRACT

In this paper, we present a parameter expansion method for two-point nonlinear singularly perturbed boundary value problem for second order ordinary differential equation. Newton linearisation scheme is used to linearise the nonlinear problem.

Keywords: Boundary Value Problems, Newton Linearisation Scheme

I. INTRODUCTION

Consider a nonlinear singularly perturbed second order boundary value problem $G_{\varepsilon y}$ defined by

$$G_{\varepsilon y} = \varepsilon y''(x) - f(x, y, y', \varepsilon)\, y'(x) = 0, \qquad a \le x \le b \tag{1}$$

$$y(a,\varepsilon) = \alpha \tag{2}$$

$$y(b,\varepsilon) = \beta \tag{3}$$

for small positive values of the parameter ε, satisfying $0 \le \varepsilon \le \varepsilon_0$ for some ε_0, while a, b, ϕ, β are independent of ε_0.

The Newton's scheme for Taylor's series expansion given by

$$G + \Delta y \frac{\partial G}{\partial y} + \Delta y' \frac{\partial G}{\partial y'} + \Delta y'' \frac{\partial G}{\partial y''} = 0 \tag{4}$$

will be used throughout this work.

Thus, from (1), we obtain the following

$$\frac{\partial G}{\partial y} = -f_y y' \;;\quad \frac{\partial G}{\partial y'} = -\,(f_{y'})y' - f \;;\quad \frac{\partial G}{y''} = \varepsilon \tag{5}$$

Substituting (5) into (4), we have

$$G_k + \Delta y_k \left(-y_k \frac{\partial f_{yk}}{\partial y_k} \right) + \Delta y'_k \left(-(f_k - y'_k \frac{\partial f_k}{\partial y'_k} \right) + \Delta y''_k(\varepsilon) = 0 \tag{6}$$

where $\Delta y^{(j)}{}_k(x) = y^{(j)}{}_{k+1}(x) - y^{(j)}{}_k(x)$

The Newton's linearisation leads to the use of the following iteration;

$$\varepsilon y''_{k+1}(x) - \left(f_k + y'_k(x) \frac{\partial f(x)}{\partial y'_k} \right) y'_{k+1}(x) - \left(y'_k \frac{\partial f_{yk}}{\partial y_k} \right) y_{k+1}(x) = \varepsilon y'_k(x) - y_k(x) y'_k(x) \frac{\partial f_{yk}}{\partial y_k} - y'_k(x) \frac{\partial f}{\partial y_k} - G_k \tag{7}$$

$$y_{k+1}(a,\varepsilon) = \phi \tag{8}$$

$$y_{k+1}(b,\varepsilon) = \beta \tag{9}$$

The Method of Parameter Expansion

The method seeks an asymptotic expansion based on the idea of Okoroet al [1], which converts the singularly perturbed problem to a system of ordinary differential equations for which the solutions are relatively easier to obtain. The ordinary differential equations are reduced to algebraic equations using the perturbed collocation method described in [1]. In order to solve (7) with this method, we seek a patched solution in two regions, namely the boundary layer region I_{BL} and outside the boundary layer region I_{OBL} where

$$I = \{x : G \le x \le b\} = I_{OBL}\, I_{BL}$$

Without any loss of generality, we set

$$I_{OBL} = \{x : a_1 \le x \le b_1\}$$
$$I_{BL} = \{x : a_2 \le x \le b_2\}$$

Where $a < a_1 < b_1 = a_2 < b_2 = b$

In the region I_{OBL}, we seek a smooth collocation solution of the form $y_{1,N}(x)$ and in the region IBL, we seek the parameter expansion $y_{2,\mathrm{M}}(x,\varepsilon)$. In the smooth solution, let,

$$y_{1,N}(x) = \sum^{N} a_1 x^{i-1}, \quad a_1 \le x \le b_1 \qquad (10)$$

satisfies exactly the slightly perturbed collocation equations

$$\varepsilon y''_{N,k+1}(x_i) - \left(f_k + y'_{N,k+1}(x_i)\frac{\partial f)}{\partial y_k}\right) y'_{N,k+1}(x_i) - \left(y'_{N,k}(x_i)\frac{\partial f_{yNk}}{\partial y_{N,k}}\right) y_{N,k+1}(x_i) =$$
$$\varepsilon y''_{N,k+1}(x) - y_{N,k}(x_i) y'_{N,k}(x)\frac{\partial f_{yNk}}{\partial y_{N,k}} - y'_{N,k}(x)\left(f_k + y'_{N,k}(x)\frac{\partial f}{\partial y_k}\right) G_k + H_N(x_i)$$

Where

$$x_i = a_i + \frac{(b_1 - a_1)^i}{N+1}; \quad i = 1, 2, \ldots, N \qquad (12)$$

$$H_N(x_i) = \sum_{K=1}^{2} t_{1,N} T^*_{N-K}(x_i); \quad a \le x \le b \qquad (13)$$

And

$$T^*_N(x_i) = T_N(2x_i - 1); \quad a \le x_i \le b \qquad (14)$$

is the shifted Chebyshev polynomial, and $t_{1,k}(k = 1, 2)$ are arbitrary constants to be determined

and

$T_N(x) = \cos[N \arccos(x)]; N \ge 0$, [-1, 1]
The zeros of $T^*_N(x)$ are given by

$$x_j = \frac{1}{2}\left((a+b) - (a-b)\cos\left((2j-1)\frac{\pi}{2N}\right)\right); \quad j = 1, \ldots, N \cdot$$

Also, $y_{1,N}(x)$ must satisfy the arbitrary conditions

$$y_{1,N}(\mathrm{a}, \varepsilon) = \phi \qquad (15)$$

and

$$y_{1,N}(\mathrm{b}, \varepsilon) = y_{2,\mathrm{M}}(\mathrm{a}_2, \varepsilon) \qquad (16)$$

The Chebyshev perturbation (14) is well-known to yield an accurate approximation.

On substituting (10) in (11) we obtain N collocation equations. Two extra equations are obtain using (15) and (16).

Altogether, we have (N + 2) collocation equations which give the unique values of $a_0, a_1, \ldots, a_N, \mu_{1,1}$ and $\mu_{1,2}$. Also, inside the boundary layer region I_{BL}, we seek a uniform valid parameter expansion in the form.

$$y_{2,N}(\mathrm{x}, \varepsilon) = \sum_{j=1}^{N} g_j(x)\varepsilon^{j-1}; \quad a_2 \le x \le b_2 \qquad (17)$$

which satisfies the following perturbed two-dimensional form of equation (7)

$$\varepsilon \frac{\partial^2 y_{2,M}(x,\varepsilon)}{\partial x^2} + p(x)\frac{\partial y_{2,M}(x,\varepsilon)}{\partial x} + q(x)y_{2,M}(x,\varepsilon) + G_1(x,\varepsilon) = H_{2,M}(\mathrm{x},\varepsilon) \qquad (18)$$

where

$$p(x) = -\left(f_k\, y'_k(x)\frac{\partial f}{\partial y_k}\right); \quad \mathrm{q}(x) = -\left(y'_k(x)\frac{\partial f_{yk}}{\partial y_k}\right)$$

$$G_1(x,\varepsilon) = y''_k(x) + y_k(x)y'_k(x)\frac{\partial f_{yk}}{\partial y_k} - y'_k(x)\left(f_k + y'_k(x)\frac{\partial f}{\partial y_k}\right) - G_k$$

$$H_{2,M}(x,\varepsilon) = \sum_{k=0}^{2} \mu_{2,1} T^*_{M-1}(\varepsilon); \quad a_2 \le x \le b_2$$

Also, $y_{2,\mathrm{M}}(\mathrm{x},\varepsilon)$ must satisfy the following conditions

$$\left.\begin{aligned} y_{2,\mathrm{M}}(\mathrm{a}_2,\varepsilon) &= y_{1,\mathrm{N}}(\mathrm{b}_1) \\ y_{2,\mathrm{M}}(\mathrm{b}_2,\varepsilon) &= \beta \end{aligned}\right\} \qquad (19)$$

and

$$\left.\begin{aligned} y_{2,\mathrm{M}}(\mathrm{x},1) &= \phi(\mathrm{x}) \\ y_{2,\mathrm{M}}(\mathrm{x},0) &= \varepsilon(\mathrm{x}) \end{aligned}\right\} \qquad (20)$$

where $\phi(\mathrm{x})$ and $\varepsilon(\mathrm{x})$ are obtained from (1) when $\varepsilon = 1$ and when $\varepsilon = 0$ respectively.

Collocating equation (19) at points ε_1, we obtain

$$\varepsilon_1 \frac{\partial^2 y_{2,M}(x,\varepsilon_1)}{\partial x^2} + p(x)\frac{\partial y_{2,M}(x,\varepsilon_1)}{\partial x} + q(x)y_{2,M}(x,\varepsilon_1) + G_1(x,\varepsilon_1) = H_{2,M}(\mathrm{x},\varepsilon_1) \qquad (21)$$

where

$$\varepsilon_i = \frac{i}{M+2}; \quad \mathrm{I} = 1, 2, \ldots, \mathrm{M}+1 \qquad (22)$$

Thus, we obtain (M + 1) second order ordinary differential equations in (M + 3) unknown functions, $g_1(x)$, $g_2(x)$, $g_3(x)$, . . . , $g_M(x)$, µ2,1 and $\mu_{2,2}$.

The arbitrary µ-functions are then eliminated to give a set of (M - 2) second order ordinary differential equations. Two extra equations are obtained using (19). Altogether, we have M second order ordinary differential equation. The M second order ordinary differential equations are then perturbed and collocated in the same manner as in (11). Equations (20) are satisfied at the Chebyshev points x_i (i = 1, 2, 3, . . . , M). These equations together with (12), (15) and (16) give the values of the constants a_i, g_{ij} (i = 1, 2, . . . , N ; j = 1, 2, . . . , M) for the required approximation.

$$y_{1,N}(\mathrm{x}) = \sum_{j=1}^{N} a_1 x^{i-1}; \quad a_1 \le x \le b_1 \tag{23}$$

$$y(x) \approx y_N(x) = y_{2,\mathrm{M}}(\mathrm{x},\varepsilon) = \sum_{j=1}^{M} g_{ji} x^{i-1} \varepsilon^{j-1}\ ; \quad a_2 \le x \le b_2$$

A Worked Example consider the nonlinear second order bvp

$$\varepsilon y''(x) - y(x)y'(x) = 0, \quad -1 \le x \le 1$$

With the boundary conditions

$$y(-1) = -\tanh\left(\frac{4}{\varepsilon}\right) \quad \text{and} \quad y(1) = \tanh\left(\frac{4}{\varepsilon}\right)$$

The analytical solution is given by

$$y(\mathrm{x}) = \tanh\left(\frac{4x}{\varepsilon}\right)$$

The Newton's iterates using (7) on (5) are given by

$$\varepsilon y''_{N,K+1}(x,\varepsilon) - y_{N,K}(x,\varepsilon)y'_{N,K+1}(x,\varepsilon) - y_{N,K}(x,\varepsilon)y_{N,K}(x,\varepsilon) = -y_{N,K}(x,\varepsilon)y'_{N,K}(x,\varepsilon); \quad x \in [-1,1]$$

and

$$y_{N,K+1}(-1,\ \varepsilon) = -\tanh\left(\frac{4}{\varepsilon}\right)$$

$$y_{N,K+1}(1,\ \varepsilon) = \tanh\left(\frac{4}{\varepsilon}\right)$$

For K = 0, the initial approximation used is

$$y_{N,K}(\mathrm{x}) = \tanh\left(\frac{4x}{\varepsilon}\right)$$

Table 1: Error Estimates for Case N = 5, M = 4

ε	Standard collocation tau method	Parameter Expansion Method
10^{-2}	1.516×10^{1}	1.516×10^{1}
10^{-3}	1.824×10^{-2}	2.053×10^{-2}
10^{-4}	1.939×10^{-3}	5.281×10^{-3}
10^{-5}	2.226×10^{-3}	2.172×10^{-3}
10^{-6}	2.251×10^{-4}	7.399×10^{-4}
10^{-7}	2.283×10^{-4}	4.052×10^{-4}
10^{-8}	2.487×10^{-5}	2.459×10^{-5}
10^{-9}	2.842×10^{-5}	2.725×10^{-5}

II. CONCLUSION

The numerical results show that the accuracy of the proposed method when compared with the standard collocation tan method improves as ε tends to zero.

III. REFERENCES

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