Existence of Nonoscillatory Solutions of First-Order Neutral Difference Equations

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ABSTRACT

In this paper, we study the existence of nonoscillatory solution of first-order neutral difference equations with delay and advance terms. Some sufficient conditions for the existence of positive solutions are obtained. Banach contraction principle is used in the proofs of the results.

Keywords and Phrases: Difference Equations, Nonoscillation, Positive Solutions, Banach Contraction Principle.

I. INTRODUCTION

In this paper, we consider a first-order neutral difference equation

\[ \Delta \left[ x(n) + P_1(n) x(n - \tau_1) + P_2(n) x(n + \tau_2) \right] + Q_1(n) x(n - \sigma_1) - Q_2(n) x(n + \sigma_2) = 0 \]  

(1.1)

where \( P_1, P_2 \in C([t_0, \infty), R) \), \( Q_1, Q_2 \in C([t_0, \infty), [0, \infty)) \), \( \tau_1, \tau_2 > 0 \) and \( \sigma_1, \sigma_2 \geq 0 \).

We present some new criteria for the existence of nonoscillatory solutions of the First Order Neutral Difference Equation (1.1). Recently, the existence of nonoscillatory solutions of neutral difference equations has been investigated by many authors, see [3, 6, 7, 10, 11] and the references contained therein. There have been several books on the subject of qualitative properties of neutral difference equations [1, 2, 5].

Let \( m = \max \{ \tau_1, \sigma_1 \} \). A solution of the difference equation (1.1) is called eventually positive if there exists a positive integer \( n_0 \) such that \( x(n) > 0 \) for \( n \in N(n_0) \).

If there exists a positive integer \( n_0 \) such that \( x(n) < 0 \) for \( n \in N(n_0) \), then (1.1) is called eventually negative.

The solution of the difference equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

This paper deals with the discrete version of the equation discussed in [9]. The following important theorem is needed in the proof of main results.

Theorem 1.1 (Banach’s Contraction Mapping Principle). A contraction mapping on a complete metric space has exactly one fixed point.

II. EXISTENCE OF POSITIVE BOUNDED SOLUTIONS

We shall show that an operator \( S \) satisfies the conditions for the contraction mapping principle by considering different cases for the ranges of the coefficients \( P_1(n) \) and \( P_2(n) \).

Theorem 2.1. Assume that

\[ 0 \leq P_1(n) \leq p_1 < 1, \]

\[ 0 \leq P_2(n) \leq p_2 < 1 - p_1 \]

and

\[ \sum_{s=t_0}^{\infty} Q_1(s) < \infty, \sum_{s=t_0}^{\infty} Q_2(s) < \infty \]  

(2.1)

Then (1.1) has a bounded non-oscillatory solution.

Proof. Because of (2.1) we can choose \( n_t \geq n_0 \),
\[ n_i \geq n_0 + \max \{ \tau_i, \sigma_i \} \]  
(2.2)

Sufficiently large such that
\[ \sum_{i=0}^{\infty} Q_i(s) \leq \frac{M_2 - \alpha}{M_2}, \quad n \geq n_i, \]  
(2.3)
\[ \sum_{i=0}^{\infty} Q_i(s) \leq \frac{\alpha - (p_1 + p_2) M_2}{M_2}, \quad n \geq n_i. \]  
(2.4)

where \( M_1 \) and \( M_2 \) are positive constants such that
\( (p_1 + p_2) M_2 + M_1 < M_2 \) and
\[ \alpha \in \left( \left( p_1 + p_2 \right) M_2 + M_1, M_2 \right). \]

Let \( l_n^\infty \) be the set of all real sequence with the norm
\[ \|x\| = \sup \{ |x(n)| \} < \infty \]. Then \( l_n^\infty \) is a Banach space. We define a closed, bounded and convex subset \( \Omega \) of \( l_n^\infty \) as follows
\[ \Omega = \{ x \in l_n^\infty : M_1 \leq x(n) \leq M_2, \quad n \geq n_0 \}. \]

Define a mapping \( S : \Omega \to l_n^\infty \) as follows
\[ (Sx)(n) = \left\{ \begin{array}{ll}
\alpha - P(n)x(n - \tau_i) - P_1(n)x(n + \tau_i) \\
\sum_{i=n}^{\infty} \left[ Q_i(s) x(s - \sigma_i) - Q_i(s) x(s + \sigma_i) \right], \quad n \geq n_i,
\end{array} \right. \]
\[ (Sx)(n_i), \quad n_i \leq n \leq n_i. \]

Obviously \( Sx \) is continuous. For \( n \geq n_i \) and \( x \in \Omega \), from (2.3) and (2.4) respectively, it follows that
\[ (Sx)(n) \leq \alpha + \sum_{i=n}^{\infty} Q_i(s) x(s - \sigma_i) \]
\[ = \alpha + M_2 \sum_{i=n}^{\infty} Q_i(s) \]
\[ = \alpha + M_2 \left( \frac{M_2 - \alpha}{M_2} \right) \]
\[ (Sx)(n) \leq M_2 \]

Also we have
\[ (Sx)(n) \geq \alpha - P(n)x(n - \tau_i) - P_1(n)x(n + \tau_i) - \sum_{i=n}^{\infty} Q_i(s) x(s + \sigma_i) \]
\[ \geq \alpha - p_1 M_2 - p_2 M_2 - \sum_{i=n}^{\infty} Q_i(s) \]
\[ = \alpha - p_1 M_2 - p_2 M_2 - M_1 \left( \frac{\alpha - (p_1 + p_2) M_2}{M_2} \right) \]
\[ (Sx)(n) \geq M_1 \]

Hence
\[ M_1 \leq (Sx)(n) \leq M_2 \] for \( n \geq n_i \).

Thus we have proved that \( (Sx)(n) \in \Omega \) for any \( n \in \Omega \).

This means that \( \Omega \subset \Omega \). To apply contraction mapping principle, we shall show \( S \) is a contraction mapping on \( \Omega \). Thus, \( x_1, x_2 \in \Omega \) and \( n \geq n_i \),
\[ \| (Sx_1)(n) - (Sx_2)(n) \| \\
= \left| \alpha - P(n)x_1(n - \tau_i) - P_1(n)x_1(n + \tau_i) - \sum_{i=n}^{\infty} \left[ Q_i(s) x_1(s - \sigma_i) - Q_i(s) x_1(s + \sigma_i) \right] \right| \]
\[ + \left| \alpha - P(n)x_2(n - \tau_i) - P_1(n)x_2(n + \tau_i) - \sum_{i=n}^{\infty} \left[ Q_i(s) x_2(s - \sigma_i) - Q_i(s) x_2(s + \sigma_i) \right] \right| \]
\[ \leq \left| \alpha - P(n)x_1(n - \tau_i) - P_1(n)x_1(n + \tau_i) \right| + \left| \alpha - P(n)x_2(n - \tau_i) - P_1(n)x_2(n + \tau_i) \right| \\
+ \sum_{i=n}^{\infty} \left| Q_i(s) \left| x_1(s - \sigma_i) - x_2(s - \sigma_i) \right| \right| \\
+ \sum_{i=n}^{\infty} \left| Q_i(s) \left| x_2(s - \sigma_i) - x_2(s + \sigma_i) \right| \right| \\
\leq p_1 \| x_1 - x_2 \| + p_2 \| x_1 - x_2 \| + \sum_{i=n}^{\infty} Q_i(s) \| x_1 - x_2 \| \\
+ \sum_{i=n}^{\infty} Q_i(s) \| x_1 - x_2 \| \\
= \left( p_1 + p_2 + \sum_{i=n}^{\infty} Q_i(s) + \sum_{i=n}^{\infty} Q_i(s) \right) \| x_1 - x_2 \| \\
= \left( p_1 + p_2 + \frac{M_2 - \alpha}{M_2} \right) \| x_1 - x_2 \| \\
= \frac{M_2 - M_1}{M_2} \left( \| x_1 - x_2 \| \right) \]

where \( \lambda_i = 1 - \frac{M_1}{M_2} \). This implies that
\[ \| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda_i \| x_1 - x_2 \|. \]

Thus we have proved that \( S \) is a contraction mapping on \( \Omega \). In fact, \( x_1, x_2 \in \Omega \) and \( n \geq n_i \), we have
\[ \| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda_i \| x_1(n - \tau_i) - x_2(n - \tau_i) \| \leq \lambda_i \| x_1 - x_2 \|. \]

Since \( 0 < \lambda_i < 1 \), we conclude that \( S \) is a contraction mapping on \( \Omega \). Thus \( S \) has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.
Theorem 2.2. Assume that $0 \leq P_1(n) \leq p_1<1,$

$p_1-1 < p_2 \leq P_1(n) \leq 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

**Proof.** Because of (2.1) we can choose $n_1 \geq n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{i=0}^{\infty} Q_i(s) \leq \frac{(1+p_2)N_2-\alpha}{N_2}, \quad n \geq n_1, \quad (2.5)$$

$$\sum_{i=0}^{\infty} Q_i(s) \leq \frac{\alpha - p_1 N_2 - N_1}{N_2}, \quad n \geq n_1, \quad (2.6)$$

where $N_1$ and $N_2$ are positive constants such that

$N_1 + p_1 N_2 < (1 + p_2) N_2$ and $\alpha \in (N_1 + p_1 N_2, (1 + p_2) N_2)$.

Let $l_n^\infty$ be the set of all real sequence with the norm

$$\|x\| = \sup \{x(n) : \|x\| < \infty \}.$$ Then $l_n^\infty$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $l_n^\infty$ as follows

$$\Omega = \{x \in l_n^\infty : N_1 \leq x(n) \leq N_2, \quad n \geq n_0 \}.$$

Define a mapping $S : \Omega \to l_n^\infty$ as follows

$$S(x)(n) = \alpha - P_1(n) x(n - \tau_1) - P_2(n) x(n + \tau_2) + \sum_{i=n}^{\infty} Q_i(s) x(s - \sigma_i), \quad n \geq n_0,$$

$$S(x)(n) = \alpha - P_2(n) x(n + \tau_2) + \sum_{i=n}^{\infty} Q_i(s) x(s + \sigma_2), \quad n \geq n_0,$$

Obviously $S(x)$ is continuous. For $n \geq n_1$ and $x \in \Omega$, from (2.5) and (2.6) respectively, it follows that

$$(Sx)(n) \leq \alpha - P_2(n) x(n + \tau_2) + \sum_{i=n}^{\infty} Q_i(s) x(s - \sigma_1)$$

$$\leq \alpha - p_2 N_2 + N_2 \sum_{i=n}^{\infty} Q_i(s)$$

$$= \alpha - p_1 N_2 + N_2 \left( 1 + p_2 \right) N_2 \left( \frac{N_2 - \alpha}{N_2} \right)$$

$$(Sx)(n) \leq N_2$$

Also

$$(Sx)(n) \geq \alpha - P_1(n) x(n - \tau_1) - \sum_{i=n}^{\infty} Q_i(s) x(s + \sigma_2)$$

$$\geq \alpha - p_1 N_2 - N_2 \sum_{i=n}^{\infty} Q_i(s)$$

$$= \alpha - p_1 N_2 - N_2 \left( \frac{\alpha - p_2 N_2 - N_1}{N_2} \right)$$

$$(Sx)(n) \geq N_1$$

Hence

$$N_1 \leq (Sx)(n) \leq N_2 \quad \text{for} \quad n \geq n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This means that $S \Omega \subset \Omega$. To apply contraction mapping principle, we shall show $S$ is a contraction mapping on $\Omega$. Thus $x_1, x_2 \in \Omega$ and $n \geq n_1$,

$$\| (Sx_1)(n) - (Sx_2)(n) \|$$

$$\leq P_1(n) \| x_1(n - \tau_1) - x_2(n - \tau_1) \|$$

$$+ P_2(n) \| x_1(n + \tau_2) - x_2(n + \tau_2) \|$$

$$+ \sum_{i=n}^{\infty} Q_i(s) \| x_1(s - \sigma_i) - x_2(s - \sigma_i) \|$$

$$+ \sum_{i=n}^{\infty} Q_i(s) \| x_1(s + \sigma_2) - x_2(s + \sigma_2) \|$$

$$\leq p_1 \| x_1 - x_2 \| + p_2 \| x_1 - x_2 \| + \sum_{i=n}^{\infty} Q_i(s) \| x_1 - x_2 \|$$

$$+ \sum_{i=n}^{\infty} Q_i(s) \| x_1 - x_2 \|$$

$$= \left( p_1 - p_2 + \sum_{i=n}^{\infty} Q_i(s) + \sum_{i=n}^{\infty} Q_i(s) \right) \| x_1 - x_2 \|$$

$$= \left( p_1 - p_2 + \left( 1 + p_2 \right) N_2 - \alpha + \alpha - p_1 N_2 - N_1 \right) \| x_1 - x_2 \|$$

$$= \frac{N_2 - N_1}{N_2} \| x_1 - x_2 \|$$

$$= \lambda_2 \| x_1 - x_2 \|$$

where $\lambda_2 = 1 - \frac{N_1}{N_2}$. This implies that

$$\| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda_2 \| x_1 - x_2 \|.$$

Thus we have proved that $S$ is a contraction mapping on $\Omega$. In fact $x_1, x_2 \in \Omega$ and $n \geq n_1$ we have

$$\| (Sx_1)(n) - (Sx_2)(n) \| \leq p(n) \| x_1(n - \tau_1) - x_2(n - \tau_1) \| \leq \lambda_2 \| x_1 - x_2 \|.$$

Since $0 < \lambda_2 < 1$, $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.3. Assume that $1 < p_1 \leq P_1(n) < |p_0| < \infty$,

$0 \leq P_2(n) \leq p_2 < p_1 - 1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

**Proof.** In view of (2.1), we can choose $n_1 \geq n_0$,

$$n_1 + \tau_1 \geq n_0 + \sigma_1 \quad (2.7)$$

Sufficiently large such that
Let $l_0^\infty$ be the set of all real sequences with the norm\n$$\|x\| = \sup_n |x(n)| < \infty.$$ Then $l_0^\infty$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $l_0^\infty$ as follows\n$$\Omega = \left\{ x \in l_0^\infty : M_3 \leq x(n) \leq M_4 , n \geq n_1 \right\}.$$\n
Define a mapping $S : \Omega \rightarrow l_0^\infty$ as follows\n\[
(Sx)(n) = \frac{1}{P_1(n + \tau_i)} \left( \alpha - x(n + \tau_i) - P_1(n + \tau_i) x(n + \tau_i + \tau_z) \right) + \sum_{i=0}^{\infty} Q_i(s) x(s + \sigma_i), n \geq n_1, \quad \text{and} \quad (Sx)(n_1) = x(n_1), \quad n_1 \leq n \leq n_1.
\]

Obviously $Sx$ is continuous. For $n \geq n_1$ and $x \in \Omega$, from \n(2.8) and (2.9) respectively, it follows that\n\[
(Sx)(n) = \frac{1}{P_1(n + \tau_i)} \left( \alpha + \sum_{i=0}^{\infty} Q_i(s) x(s + \sigma_i) \right)
\]
\[
\leq \frac{1}{P_1} \left( \alpha + M_4 \sum_{i=0}^{\infty} Q_i(s) \right)
\]
\[
= \frac{1}{P_1} \left( \alpha + M_4 \left( \frac{p_1 M_4 - \alpha}{M_4} \right) \right)
\]
\[
(Sx)(n) \leq M_4
\]
\[
\sum_{j=n}^{\infty} Q_j(s) \leq \frac{p_1 M_4 - \alpha}{M_4}, \quad n \geq n_1.
\]

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show $S$ is a contraction mapping on $\Omega$. Thus $\lambda_1, \lambda_2 \in \Omega$ for $n \geq n_1$,
\[
\| (Sx_1)(n) - (Sx_2)(n) \|
\]
\[
\leq \frac{1}{P_1(n + \tau_i)} \left( |x_1(n + \tau_i) - x_2(n + \tau_i)| + P_1(n + \tau_i) |x_1(n + \tau_i + \tau_z) - x_2(n + \tau_i + \tau_z)| \right)
\]
\[
+ \sum_{i=0}^{\infty} Q_i(s) |x_1(s + \sigma_i) - x_2(s + \sigma_i)| + \sum_{i=0}^{\infty} Q_i(s) |x_1(s + \sigma_i) - x_2(s + \sigma_i)|
\]
\[
\leq \frac{1}{P_1(n + \tau_i)} \left( p_1 + \sum_{i=0}^{\infty} Q_i(s) + \sum_{i=0}^{\infty} Q_i(s) \right) \| x_1 - x_2 \|
\]
\[
= \frac{1}{P_1} \left( p_1 + \frac{p_1 M_4 - \alpha}{M_4} \right) \| x_1 - x_2 \|
\]
\[
= \frac{1}{P_1} \left( p_1 M_4 - \frac{p_1 M_4}{M_4} \right) \| x_1 - x_2 \|
\]
\[
= \lambda_3 \| x_1 - x_2 \|
\]

where $\lambda_3 = 1 - \frac{p_1 M_4}{p_1 M_4}$. This implies that
\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda_3 \| x_1 - x_2 \|
\]

Thus we have proved that $S$ is a contraction mapping on $\Omega$. In fact $\lambda_1, \lambda_2 \in \Omega$ and $n \geq n_1$ we have
\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \rho(n) \| x_1(n + \tau_i) - x_2(n + \tau_i) \| \leq \lambda_3 \| x_1 - x_2 \|
\]

Since $0 < \lambda_3 < 1 $ we conclude that $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

**Theorem 2.4.** Assume that $1 < p_1 \leq P_1(n) < p_0 < 1$ , $1 - p_1 < p_2 \leq P_2(n) \leq 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

**Proof.** In view of (2.1), we can choose $n_1 \geq n_0$ sufficiently large satisfying (2.7) such that
\[
\sum_{n=0}^{\infty} Q_1(s) \leq \frac{(p_1 + p_2)N_4 - \alpha}{N_4}, n \geq n_1, \tag{2.10}
\]
\[
\sum_{n=0}^{\infty} Q_2(s) \leq \frac{\alpha - p_b N_3 - N_4}{N_4}, n \geq n_1. \tag{2.11}
\]

where \( N_1 \) and \( N_4 \) are positive constants such that
\[
p_b N_3 + N_4 < (p_1 + p_2) N_4\text{ and } \alpha \in (p_b N_3 + N_4, (p_1 + p_2) N_4).\]

Let \( I^\infty_{n_0} \) be the set of all real sequences with the norm \( \|x\| = \sup \{x(n) \} < \infty \). Then \( I^\infty_{n_0} \) is a Banach space. We define a closed, bounded and convex subset \( \Omega \) of \( I^\infty_{n_0} \) as follows
\[
\Omega = \left\{ x \in I^\infty_{n_0} : N_3 \leq x(n) \leq N_4, n \geq n_0 \right\}.
\]

Define a mapping \( S : \Omega \rightarrow I^\infty_{n_0} \) as follows
\[
(Sx)(n) = \sum_{s=0}^{\infty} \left[ Q_1(s) x(s + \sigma_1) - Q_2(s) x(s + \sigma_2) \right], n \geq n_1,
\]
\[
(Sx)(n), \quad n_0 \leq n \leq n_1.
\]

Obviously \( Sx \) is continuous. For \( n \geq n_1 \) and \( x \in \Omega \), from (2.10) and (2.11) respectively, it follows that
\[
(Sx)(n) \leq \frac{1}{P_1(n + \tau_1)} \left( \alpha - p_2 (n + \tau_1) x(n + \tau_1 + \tau_2) \right)
\]
\[
+ \sum_{s=0}^{\infty} Q_2(s) x(s - \sigma_1)
\]
\[
\leq \frac{1}{P_1} \left( \alpha - p_2 N_4 + N_4 \sum_{s=0}^{\infty} Q_1(s) \right)
\]
\[
= \frac{1}{P_1} \left( \alpha - p_2 N_4 + N_4 \left( \frac{(p_1 + p_2) N_4 - \alpha}{N_4} \right) \right)
\]
\[
(Sx)(n) \leq N_4
\]

Furthermore
\[
(Sx)(n) \geq \frac{1}{P_1(n + \tau_1)} \left( \alpha - x(n + \tau_1) \right)
\]
\[
- \sum_{s=0}^{\infty} Q_2(s) x(s + \sigma_2)
\]
\[
\geq \frac{1}{P_1} \left( \alpha - N_4 - N_4 \sum_{s=0}^{\infty} Q_2(s) \right)
\]
\[
= \frac{1}{P_1} \left( \alpha - N_4 - N_4 \left( \frac{\alpha - p_b N_3 - N_4}{N_4} \right) \right)
\]
\[
(Sx)(n) \geq N_3
\]

Hence
\[
N_3 \leq (Sx)(n) \leq N_4 \text{ for } n \geq n_1 \cdot
\]

Thus we have proved that \( (Sx)(n) \in \Omega \) for any \( x \in \Omega \).

This means that \( S(\Omega) \subset \Omega \). To apply contraction mapping principle, we shall show \( S \) is a contraction mapping on \( \Omega \). Thus \( x_1, x_2 \in \Omega \) and \( n \geq n_1 \),
\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \frac{1}{P_1} \left( k_1 \| x_1 - x_2 \| - p_2 \| x_1 - x_2 \| + \sum_{s=0}^{\infty} Q_2(s) \| x_1 - x_2 \| + \sum_{s=0}^{\infty} Q_2(s) \| x_1 - x_2 \| \right)
\]
\[
\leq \frac{1}{P_1} \left( \sum_{s=0}^{\infty} Q_1(s) \| x_1 - x_2 \| + \sum_{s=0}^{\infty} Q_2(s) \| x_1 - x_2 \| \right)
\]
\[
= \frac{1}{P_1} \left( 1 - p_2 \sum_{s=0}^{\infty} Q_1(s) + \sum_{s=0}^{\infty} Q_2(s) \right) \| x_1 - x_2 \|
\]
\[
= \frac{1}{P_1} \left( 1 - p_2 + \left( p_1 + p_2 \right) N_4 - \alpha \right) \| x_1 - x_2 \|
\]
\[
\| x_1 - x_2 \|
\]
\[
= \frac{1}{P_1} \left( p_b N_3 - N_4 \right) \| x_1 - x_2 \|
\]
\[
= \lambda \| x_1 - x_2 \|
\]

where \( \lambda = 1 - \frac{p_b N_3}{p_1 N_4} \). This implies that
\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda \| x_1 - x_2 \|.
\]

Thus we have proved that \( S \) is a contraction mapping on \( \Omega \). In fact \( x_1, x_2 \in \Omega \) and \( n \geq n_1 \) we have
\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \rho(n) \| x_1(n - \tau_1) - x_2(n - \tau_1) \| \leq \lambda \| x_1 - x_2 \|.
\]

Since \( 0 < \lambda < 1 \), \( S \) is a contraction mapping on \( \Omega \). Thus \( S \) has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.
**Theorem 2.5.** Assume that \(-1 < p_i \leq P_i (n) \leq 0\), 
\(0 \leq P_i (n) \leq p_i < 1 + p_i\) and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

**Proof.** From (2.1), we can choose \(n_i \geq n_0\) sufficiently large satisfying (2.2) such that

\[
\sum_{s=1}^{\infty} Q_i (s) \leq \left( \frac{1 + p_i}{M_6} - \alpha \right) n \geq n_i, \quad \text{(2.12)}
\]

\[
\sum_{s=1}^{\infty} Q_2 (s) \leq \frac{\alpha - p_2 M_6 - M_5}{M_6}, n \geq n_i. \quad \text{(2.13)}
\]

where \(M_2\) and \(M_6\) are positive constants such that \(M_2 + p_2 M_6 < (1 + p_1) M_6\) and

\[
\alpha \in \left( M_2 + p_2 M_6, (1 + p_1) M_6 \right).
\]

Let \(l_{n_0}^\infty\) be the set of all real sequence with the norm \(\|x\| = \sup_n |x(n)| < \infty\). Then \(l_{n_0}^\infty\) is a Banach space. We define a closed, bounded and convex subset \(\Omega\) of \(l_{n_0}^\infty\) as follows

\[
\Omega = \{ x \in l_{n_0}^\infty : M_5 \leq x(n) \leq M_6, n \geq n_0 \}.
\]

Define a mapping \(S : \Omega \to l_{n_0}^\infty\) as follows

\[
(Sx)(n) = \left\{ \begin{array}{ll}
\alpha - P_1 (n) x(n - \tau) - P_1 (n) x(n + \tau), & n \leq n_0, \\
\sum_{s=1}^{\infty} Q_1 (s) x(s - \sigma) - Q_1 (s) x(s + \sigma), & n \geq n_1,
\end{array} \right.
\]

Obviously \(Sx\) is continuous. For \(n \geq n_1\) and \(x \in \Omega\), from (2.12) and (2.13) respectively, it follows that

\[
(Sx)(n) \leq \alpha - P_1 (n) x(n - \tau) + \sum_{s=1}^{\infty} Q_1 (s) x(s - \sigma)
\leq \alpha - p_1 M_6 + M_6 \sum_{s=1}^{\infty} Q_1 (s)
= \alpha - p_1 M_6 + M_6 \left( \frac{1 + p_1}{M_6} - \alpha \right)
= M_6
\]

\[
(Sx)(n) \leq M_6.
\]

Also we have

\[
(Sx)(n) \geq \alpha - P_2 (n) x(n + \tau) - \sum_{s=1}^{\infty} Q_2 (s) x(s + \sigma)
\geq \alpha - p_2 M_6 - M_6 \sum_{s=1}^{\infty} Q_2 (s)
= \alpha - p_2 M_6 - M_6 \left( \frac{\alpha - p_2 M_6 - M_5}{M_6} \right)
= M_5
\]

\[
(Sx)(n) \geq M_5.
\]

Hence

\[
M_5 \leq (Sx)(n) \leq M_6, \quad n \geq n_1.
\]

Thus we have proved that \((Sx)(n) \in \Omega\) for any \(x \in \Omega\).

This means that \(S \Omega \subset \Omega\). To apply contraction mapping principle, it remains to show that \(S\) is a contraction mapping on \(\Omega\). Thus \(x_1, x_2 \in \Omega\) and

\[
\|x_1 - x_2\| \leq -p_1 \|x_1 - x_2\| + p_2 \|x_1 - x_2\| + \sum_{s=1}^{\infty} Q_1 (s) \|x_1 - x_2\|
+ \sum_{s=1}^{\infty} Q_2 (s) \|x_1 - x_2\|
= \left[ -p_1 + p_2 + \sum_{s=1}^{\infty} Q_1 (s) + \sum_{s=1}^{\infty} Q_2 (s) \right] \|x_1 - x_2\|
= \left[ -p_1 + p_2 + \frac{1 + p_1}{M_6} - \alpha + \frac{\alpha - p_2 M_6 - M_5}{M_6} \right]
\|x_1 - x_2\|
= \frac{M_6 - M_6 - \alpha}{M_6} \|x_1 - x_2\|
= \lambda_5 \|x_1 - x_2\|
\]

where \(\lambda_5 = 1 - \frac{M_6}{M_6}\). This implies that

\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \lambda_5 \|x_1 - x_2\|.
\]

Thus we have proved that \(S\) is a contraction mapping on \(\Omega\). In fact \(x_1, x_2 \in \Omega\) and \(n \geq n_1\) we have

\[
\| (Sx_1)(n) - (Sx_2)(n) \| \leq \rho(n) |x_1(n - \tau) - x_2(n - \tau)| \leq \lambda_5 \|x_1 - x_2\|.
\]

Since \(0 < \lambda_5 < 1\), we conclude that \(S\) is a contraction mapping on \(\Omega\). Thus \(S\) has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.
Theorem 2.6. Assume that \(-1 < p_i \leq P_i(n) \leq 0\), \(-1 - p_i < p_2 \leq P_2(n) \leq 0\) and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. From (2.1), we can choose \(n_i \geq n_0\) sufficiently large satisfying (2.2) such that
\[
\sum_{i=n}^{\infty} Q_i(s) \leq \left(1 + p_i + p_2\right) N_6 - \alpha, \quad n \geq n_i, \quad (2.14)
\]
\[
\sum_{i=n}^{\infty} Q_2(s) \leq \frac{\alpha - N_5}{N_6}, \quad n \geq n_i \quad (2.15)
\]
where \(N_5\) and \(N_6\) are positive constants such that
\[
N_5 < (1 + p_i + p_2) N_6 \quad \text{and} \quad \alpha \in \left(N_5, (1 + p_i + p_2) N_6\right).
\]
Let \(l^n_o\) be the set of all real sequence with the norm \(\|x\| = \sup x(n) < \infty\). Then \(l^n_o\) is a Banach space. We define a closed, bounded and convex subset \(\Omega\) of \(l^n_o\) as follows
\[
\Omega = \left\{x \in l^n_o : N_5 \leq x(n) \leq N_6, \quad n \geq n_i \right\}.
\]
Define a mapping \(S : \Omega \to l^n_o\) as follows
\[
(Sx)(n) = \begin{cases} \alpha - P_1(n) x(n - \tau_i) - P_2(n) x(n + \tau_i) \\
\sum_{j=1}^{\infty} \left[ Q_1(s) x(s - \sigma_i) - Q_2(s) x(s + \sigma_i) \right], n \geq n_i, \\
(Sx)(n), \quad n_i \leq n \leq n_i.
\end{cases}
\]
Obviously \(Sx\) is continuous. For \(n \geq n_i\) and \(x \in \Omega\), from (2.14) and (2.15) respectively, it follows that
\[
(Sx)(n) \leq \alpha - p_1 N_6 - p_2 N_6 + N_6 \sum_{i=n}^{\infty} Q_i(s)
\]
\[
= \alpha - p_1 N_6 - p_2 N_6 + N_6 \left(1 + p_i + p_2\right) N_6 - \alpha
\]
\[
= \alpha - p_1 N_6 - p_2 N_6 + N_6 \left(1 + p_i + p_2\right) N_6 - \alpha
\]
\[
(Sx)(n) \leq N_6
\]
Furthermore
\[
(Sx)(n) \geq \alpha - N_6 \sum_{i=n}^{\infty} Q_2(s)
\]
\[
\geq \alpha - N_6 \sum_{i=n}^{\infty} Q_2(s)
\]
\[
= \alpha - N_6 \left(\frac{\alpha - N_5}{N_6}\right)
\]
\[
(Sx)(n) \geq N_5
\]
Hence
\[
N_5 \leq (Sx)(n) \leq N_6 \quad \text{for} \quad n \geq n_i.
\]
Thus we have proved that \((Sx)(n) \in \Omega\) for any \(x \in \Omega\). This means that \(\Omega \subset \Omega\). To apply contraction mapping principle, we shall show \(S\) is a contraction mapping on \(\Omega\). Thus \(x_1, x_2 \in \Omega\) and \(n \geq n_i\),
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \leq -p_1 \|x_1 - x_2\| - p_2 \|x_1 - x_2\| + \sum_{i=n}^{\infty} Q_i(s) \|x_1 - x_2\|
\]
\[
= \|p_1 - p_2 + (1 + p_i + p_2) N_6 - \alpha + \alpha - N_5\| N_6
\]
\[
\|x_1 - x_2\|
\]
\[
= \frac{N_6 - N_5}{N_6} \|x_1 - x_2\|
\]
where \(\lambda_n = 1 - \frac{N_5}{N_6}\). This implies that
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \leq \lambda_n \|x_1 - x_2\|
\]
Thus we have proved that \(S\) is a contraction mapping on \(\Omega\). In fact \(x_1, x_2 \in \Omega\) and \(n \geq n_i\) we have
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \leq p(n) \|x_1(n - \tau_i) - x_2(n - \tau_i)\| \leq \lambda_n \|x_1 - x_2\|
\]
Since \(0 < \lambda_n < 1\), \(S\) is a contraction mapping on \(\Omega\). Thus \(S\) has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.7. Assume that \(-\infty < p_i \leq P_1(n) < p_i < -1\), \(0 \leq P_2(n) \leq p_2 < -p_1 - 1\) and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. In view of (2.1), we can choose \(n_i \geq n_0\) sufficiently large satisfying (2.7) such that
\[
\sum_{i=n}^{\infty} Q_i(s) \leq \frac{p_i M_5 + \alpha}{M_6}, \quad n \geq n_i \quad (2.16)
\]
\[
\sum_{i=n}^{\infty} Q_2(s) \leq \frac{-p_1 - 1 - p_2}{M_6}, \quad n \geq n_i \quad (2.17)
\]
where \(M_5\) and \(M_6\) are positive constants such that
\(-p_i M_5 < (-p_1 - 1 - p_2) M_6\) and
\[
\alpha \in (-p_i M_5, (-p_1 - 1 - p_2) M_6).
\]
Let $l_n^\omega$ be the set of all real sequence with the norm
$$\|x\| = \sup |x(n)| < \infty.$$ Then $l_n^\omega$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $l_n^\omega$ as follows
$$\Omega = \{ x \in l_n^\omega : M_x \leq x(n) \leq M_x, \ n \geq n_0 \}.$$ Define a mapping $S : \Omega \rightarrow l_n^\omega$ as follows
$$(Sx)(n) = \left\{ \begin{array}{cl}
\frac{-1}{P_1(n + \tau)} \left( \alpha + x(n + \tau) + P_1(n + \tau)x(n + \tau + \tau) \right) \\
+ \sum_{i=n}^{\infty} Q_i(s) x(s - \sigma) \\
+ \sum_{i=n}^{\infty} Q_i(s) x(s + \sigma) \right), & n \geq n_0,
\end{array} \right.$$ where $\Omega$ is a Banach space. Then $S$ is a contraction mapping on $\Omega$. In order to apply contraction mapping principle, we shall show $S$ is a contraction mapping on $\Omega$. Thus if $x_1, x_2 \in \Omega$ and $n \geq n_1$, we have
$$\| (Sx_1)(n) - (Sx_2)(n) \| \leq -1 \frac{1}{p_1} \left( -1 + \sum_{i=n}^{\infty} Q_i(s) + \sum_{i=n}^{\infty} Q_i(s) \right) \| x_1 - x_2 \|.$$ We thus proved that $S$ is a contraction mapping on $\Omega$. Thus $S$ has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

**Theorem 2.8.** Assume that $-\infty < p_\omega \leq P_1(n) < p_1 < -1,$ $p_1 + 1 < p_2 \leq P_2(n) \leq 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

**Proof.** In view of (2.1), we can choose $n_1 \geq n_0$ sufficiently large satisfying (2.7) such that
$$\sum_{i=n}^{\infty} Q_i(s) \leq \frac{p_\omega N_7 + p_2 N_8 + \alpha}{N_8}, \quad n \geq n_1, \quad (2.18)$$
$$\sum_{i=n}^{\infty} Q_i(s) \leq \frac{-p_1 - 1) N_8 - \alpha}{N_8}, \quad n \geq n_1, \quad (2.19)$$ where $N_7$ and $N_8$ are positive constants such that $-p_\omega N_7 + p_2 N_8 < (-p_1 - 1) N_8$ and
$$\alpha \in \left( -p_\omega N_7 - p_2 N_8, (-p_1 - 1) N_8 \right).$$ Let $l_n^\omega$ be the set of all real sequence with the norm
$$\|x\| = \sup |x(n)| < \infty.$$ Then $l_n^\omega$ is a Banach space. We define a closed, bounded and convex subset $\Omega$ of $l_n^\omega$ as follows
$$\Omega = \{ x \in l_n^\omega : M_x \leq x(n) \leq M_x, \ n \geq n_0 \}.$$
Define a mapping \( S: \Omega \to \ell^\infty_{\mathbb{N}_0} \) as follows
\[
(Sx)(n) = \begin{cases}
-\frac{1}{P_1(n + \tau_1)}(\alpha + x(n + \tau_1) + P_1(n + \tau_1)x(n + \tau_1 + \tau_2) \\
- \sum_{s=n+\tau_1}^{\infty} Q_s(s)x(s-\sigma_1) - Q_s(s)x(s+\sigma_2) \end{cases}, n \geq n_0,
\]
\((Sx)(n_0), \quad n_0 \leq n \leq n_1.
\]
Obviously \( Sx \) is continuous. For \( n \geq n_0 \) and \( x \in \Omega \), from (2.19) and (2.18) respectively, it follows that
\[
(Sx)(n) \leq -\frac{1}{P_1(n + \tau_1)}(\alpha + x(n + \tau_1) + \sum_{s=n+\tau_1}^{\infty} Q_s(s)x(s+\sigma_2)) \\
\leq -\frac{1}{P_1} \left( \alpha + N_8 + N_8 \sum_{s=n}^{\infty} Q_s(s) \right) \\
= -\frac{1}{P_1} \left( \alpha + N_8 + N_8 \left( \frac{(-p_1 - 1)N_8 - \alpha}{N_8} \right) \right)
\]
\((Sx)(n) \leq N_8
\]

Furthermore
\[
(Sx)(n) \geq -\frac{1}{P_1(n + \tau_1)}(\alpha + P_1(n + \tau_1)x(n + \tau_1 + \tau_2) \\
- \sum_{s=n+\tau_1}^{\infty} Q_s(s)x(s-\sigma_1)) \\
\geq -\frac{1}{P_1} \left( \alpha + p_1 N_8 \right) - N_8 \sum_{s=n}^{\infty} Q_s(s) \\
= -\frac{1}{P_1} \left( \alpha + p_1 N_8 \right) - N_8 \left( \frac{p_1 N_8 + p_2 N_8 + \alpha}{N_8} \right)
\]
\((Sx)(n) \geq N_7
\]

Hence
\[
N_7 \leq (Sx)(n) \leq N_8 \quad \text{for} \quad n \geq n_1.
\]

Thus we have proved that \( (Sx)(n) \in \Omega \) for any \( x \in \Omega \).

This means that \( S \Omega \subset \Omega \). To apply contraction mapping principle, we shall show \( S \) is a contraction mapping on \( \Omega \). Thus \( x_1, x_2 \in \Omega \) and \( n \geq n_1 \),
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \\
\leq -\frac{1}{P_1(n + \tau_1)}\left( |x_1(n + \tau_1) - x_2(n + \tau_1)| + \sum_{s=n+\tau_1}^{\infty} Q_s(s)\|x_1(n + \tau_1) - x_2(n + \tau_1)\| \right)
\]
\[
= -\frac{1}{P_1}\left( 1 - p_1 + \sum_{s=n+\tau_1}^{\infty} Q_s(s)\|x_1(n + \tau_1) - x_2(n + \tau_1)\| \right)
\]
\[
= -\frac{1}{P_1}\left( 1 - p_1 + p_2 N_8 + p_2 N_8 + \alpha \left( (-p_1 - 1)N_8 - \alpha \right) \right)
\]
\[
\leq -\frac{1}{P_1}\left( 1 - p_1 N_8 - p_1 N_8 \right)\|x_1(n + \tau_1) - x_2(n + \tau_1)\|
\]
\[
= \lambda_0\|x_1(n + \tau_1) - x_2(n + \tau_1)\|
\]
where \( \lambda_0 = 1 - \frac{p_1 N_7}{p_1 N_8} \). This implies that
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \leq \lambda_0\|x_1(n + \tau_1) - x_2(n + \tau_1)\|
\]
Thus we have proved that \( S \) is a contraction mapping on \( \Omega \). In fact \( x_1, x_2 \in \Omega \) and \( n \geq n_1 \) we have
\[
\|(Sx_1)(n) - (Sx_2)(n)\| \leq \lambda_0\|x_1(n + \tau_1) - x_2(n + \tau_1)\| \leq \lambda_0\|x_1(n + \tau_1) - x_2(n + \tau_1)\|
\]
Since \( 0 < \lambda_0 < 1 \), \( S \) is a contraction mapping on \( \Omega \). Thus \( S \) has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

III. REFERENCES


