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Existence of Nonoscillatory Solutions of First-Order Neutral Difference Equations

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ABSTRACT

In this paper, we study the existence of nonoscillatory solution of first-order neutral difference equations with delay and advance terms. Some sufficient conditions for the existence of positive solutions are obtained. Banach contraction principle is used in the proofs of the results.

Keywords and Phrases: Difference Equations, Nonoscillation, Positive Solutions, Banach Contraction Principle.

I. INTRODUCTION

In this paper, we consider a first-order neutral difference equation

$$\begin{aligned}&\Delta\left[x(n)+P_1(n)x(n-\tau_1)+P_2(n)x(n+\tau_2)\right]\\&+Q_1(n)\,x(n-\sigma_1)-Q_2(n)\,x(n+\sigma_2)=0\end{aligned}\qquad(1.1)$$

where $P_1,P_2\in C([t_0,\infty),R)$, $Q_1,Q_2\in C([t_0,\infty),[0,\infty))$, $\tau_1,\tau_2>0$ and $\sigma_1,\sigma_2\geq 0$.

We present some new criteria for the existence of non-oscillatory solutions of the First Order Neutral Difference Equation (1.1). Recently, the existence of nonoscillatory solutions of neutral difference equations has been investigated by many authors, see [3, 6, 7, 10, 11] and the references contained therein. There have been several books on the subject of qualitative properties of neutral difference equations [1, 2, 5].

Let $m=\max\{\tau_1,\sigma_1\}$. A solution of the difference equation (1.1) is called eventually positive if there exists a positive integer n_0 such that $x(n)>0$ for $n\in N(n_0)$. If there exists a positive integer n_0 such that $x(n)<0$ for $n\in N(n_0)$, then (1.1) is called eventually negative.

The solution of the difference equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

This paper deals with the discrete version of the equation discussed in [9]. The following important theorem is needed in the proof of main results.

Theorem 1.1 (Banach's Contraction Mapping Principle). A contraction mapping on a complete metric space has exactly one fixed point.

II. EXISTENCE OF POSITIVE BOUNDED SOLUTIONS

We shall show that an operator S satisfies the conditions for the contraction mapping principle by considering different cases for the ranges of the coefficients $P_1(n)$ and $P_2(n)$.

Theorem 2.1. Assume that $0\leq P_1(n)\leq p_1<1$, $0\leq P_2(n)\leq p_2<1-p_1$ and

$$\sum_{s=n_0}^{\infty}Q_1(s)<\infty,\ \sum_{s=n_0}^{\infty}Q_2(s)<\infty\qquad(2.1)$$

Then (1.1) has a bounded non-oscillatory solution.

Proof. Because of (2.1) we can choose $n_1\geq n_0$,

$$n_1 \geq n_0 + \max\{\tau_1, \sigma_1\} \tag{2.2}$$

Sufficiently large such that

$$\sum_{s=n}^{\infty} Q_1(s) \leq \frac{M_2 - \alpha}{M_2}, \quad n \geq n_1, \tag{2.3}$$

$$\sum_{s=n}^{\infty} Q_2(s) \leq \frac{\alpha - (p_1 + p_2)M_2 - M_1}{M_2}, n \geq n_1. \tag{2.4}$$

where M_1 and M_2 are positive constants such that $(p_1 + p_2)M_2 + M_1 < M_2$ and $\alpha \in ((p_1 + p_2)M_2 + M_1, M_2)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{x \in l_{n_0}^{\infty} : M_1 \leq x(n) \leq M_2, \, n \geq n_0\right\}.$$

Define a mapping $S:\Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ +\sum_{s=n}^{\infty}\left[Q_1(s)\, x(s-\sigma_1) - Q_2(s)\, x(s+\sigma_2)\right], \; n \geq n_1, \\ (Sx)(n_1), \qquad\qquad n_0 \leq n \leq n_1. \end{cases}$$

Obviously Sx is continuous. For $n \geq n_1$ and $x \in \Omega$, from (2.3) and (2.4) respectively, it follows that

$$\begin{aligned}(Sx)(n) &\leq \alpha + \sum_{s=n}^{\infty} Q_1(s)\, x(s-\sigma_1) \\ &= \alpha + M_2 \sum_{s=n}^{\infty} Q_1(s) \\ &= \alpha + M_2\left(\frac{M_2 - \alpha}{M_2}\right) \\ (Sx)(n) &\leq M_2\end{aligned}$$

Also we have

$$\begin{aligned}(Sx)(n) &\geq \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) - \sum_{s=n}^{\infty} Q_2(s)\, x(s+\sigma_2) \\ &\geq \alpha - p_1M_2 - p_2M_2 - M_2\sum_{s=n}^{\infty} Q_2(s) \\ &= \alpha - p_1M_2 - p_2M_2 - M_2\left(\frac{\alpha - (p_1+p_2)M_2 - M_1}{M_2}\right) \\ (Sx)(n) &\geq M_1\end{aligned}$$

Hence

$$M_1 \leq (Sx)(n) \leq M_2 \text{ for } n \geq n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \geq n_1$,

$$\begin{aligned}&\left|(Sx_1)(n) - (Sx_2)(n)\right| \\ &= \Big|\alpha - P_1(n)x_1(n-\tau_1) - P_2(n)x_1(n+\tau_2) \\ &\quad +\sum_{s=n}^{\infty}\left[Q_1(s)\, x_1(s-\sigma_1) - Q_2(s)\, x_1(s+\sigma_2)\right] \\ &\quad -\Big(\alpha - P_1(n)x_2(n-\tau_1) - P_2(n)x_2(n+\tau_2) \\ &\quad +\sum_{s=n}^{\infty}\left[Q_1(s)\, x_2(s-\sigma_1) - Q_2(s)\, x_2(s+\sigma_2)\right]\Big)\Big| \\ &\leq P_1(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \\ &\quad +P_2(n)\left|x_1(n+\tau_2) - x_2(n+\tau_2)\right| \\ &\quad +\sum_{s=n}^{\infty} Q_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| \\ &\quad +\sum_{s=n}^{\infty} Q_2(s)\left|x_1(s+\sigma_2) - x_2(s+\sigma_2)\right| \\ &\leq p_1\|x_1 - x_2\| + p_2\|x_1 - x_2\| + \sum_{s=n}^{\infty} Q_1(s)\|x_1 - x_2\| \\ &\quad +\sum_{s=n}^{\infty} Q_2(s)\|x_1 - x_2\| \\ &= \left(p_1 + p_2 + \sum_{s=n}^{\infty} Q_1(s) + \sum_{s=n}^{\infty} Q_2(s)\right)\|x_1 - x_2\| \\ &= \left(p_1 + p_2 + \frac{M_2 - \alpha}{M_2} + \frac{\alpha - (p_1+p_2)M_2 - M_1}{M_2}\right) \\ &\quad \|x_1 - x_2\| \\ &= \frac{M_2 - M_1}{M_2}\left(\|x_1 - x_2\|\right) \\ &= \lambda_1\|x_1 - x_2\|\end{aligned}$$

where $\lambda_1 = 1 - \frac{M_1}{M_2}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \leq \lambda_1\|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \geq n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \leq p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \leq \lambda_1\|x_1 - x_2\|.$$

Since $0 < \lambda_1 < 1$, we conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.2. Assume that $0 \le P_1(n) \le p_1 < 1$, $p_1 - 1 < p_2 \le P_2(n) \le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. Because of (2.1) we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{(1+p_2)N_2 - \alpha}{N_2}, \quad n \ge n_1, \tag{2.5}$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{\alpha - p_1 N_2 - N_1}{N_2}, \quad n \ge n_1. \tag{2.6}$$

where N_1 and N_2 are positive constants such that $N_1 + p_1 N_2 < (1+p_2)N_2$ and $\alpha \in (N_1 + p_1 N_2, (1+p_2)N_2)$. Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{ x \in l_{n_0}^{\infty} : N_1 \le x(n) \le N_2,\ n \ge n_0 \right\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ + \sum_{s=n}^{\infty} \left[Q_1(s)\, x(s-\sigma_1) - Q_2(s)\, x(s+\sigma_2) \right], & n \ge n_1, \\ (Sx)(n_1), & n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.5) and (2.6) respectively, it follows that

$$(Sx)(n) \le \alpha - P_2(n)x(n+\tau_2) + \sum_{s=n}^{\infty} Q_1(s)\, x(s-\sigma_1)$$

$$\le \alpha - p_2 N_2 + N_2 \sum_{s=n}^{\infty} Q_1(s)$$

$$= \alpha - p_2 N_2 + N_2 \left(\frac{(1+p_2)N_2 - \alpha}{N_2} \right)$$

$$(Sx)(n) \le N_2$$

Also

$$(Sx)(n) \ge \alpha - P_1(n)x(n-\tau_1) - \sum_{s=n}^{\infty} Q_2(s)\, x(s+\sigma_2)$$

$$\ge \alpha - p_1 N_2 - N_2 \sum_{s=n}^{\infty} Q_2(s)$$

$$= \alpha - p_1 N_2 - N_2 \left(\frac{\alpha - p_1 N_2 - N_1}{N_2} \right)$$

$$(Sx)(n) \ge N_1$$

Hence

$$N_1 \le (Sx)(n) \le N_2 \text{ for } n \ge n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\begin{aligned} &\left|(Sx_1)(n) - (Sx_2)(n)\right| \\ &\le P_1(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \\ &\quad + P_2(n)\left|x_1(n+\tau_2) - x_2(n+\tau_2)\right| \\ &\quad + \sum_{s=n}^{\infty} Q_1(s)\left|x_1(s-\sigma_1) - x_2(s-\sigma_1)\right| \\ &\quad + \sum_{s=n}^{\infty} Q_2(s)\left|x_1(s+\sigma_2) - x_2(s+\sigma_2)\right| \\ &\le p_1\|x_1 - x_2\| - p_2\|x_1 - x_2\| + \sum_{s=n}^{\infty} Q_1(s)\|x_1 - x_2\| \\ &\quad + \sum_{s=n}^{\infty} Q_2(s)\|x_1 - x_2\| \\ &= \left(p_1 - p_2 + \sum_{s=n}^{\infty} Q_1(s) + \sum_{s=n}^{\infty} Q_2(s) \right)\|x_1 - x_2\| \\ &= \left(p_1 - p_2 + \frac{(1+p_2)N_2 - \alpha}{N_2} + \frac{\alpha - p_1 N_2 - N_1}{N_2} \right) \\ &\quad \|x_1 - x_2\| \\ &= \frac{N_2 - N_1}{N_2}\left(\|x_1 - x_2\|\right) \\ &= \lambda_2\|x_1 - x_2\| \end{aligned}$$

where $\lambda_2 = 1 - \dfrac{N_1}{N_2}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \le \lambda_2\|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \le p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \le \lambda_2\|x_1 - x_2\|.$$

Since $0 < \lambda_2 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.3. Assume that $1 < p_1 \le P_1(n) < p_{1_0} < \infty$, $0 \le P_2(n) \le p_2 < p_1 - 1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. In view of (2.1), we can choose $n_1 \ge n_0$,

$$n_1 + \tau_1 \ge n_0 + \sigma_1 \tag{2.7}$$

Sufficiently large such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{p_1 M_4 - \alpha}{M_4}, \quad n \ge n_1, \tag{2.8}$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{\alpha - p_{1_0} M_3 - (1+p_2) M_4}{M_4}, n \ge n_1. \tag{2.9}$$

where M_3 and M_4 are positive constants such that

$p_{1_0} M_3 + (1+p_2) M_4 < p_1 M_4$ and

$\alpha \in \left(p_{1_0} M_3 + (1+p_2) M_4, p_1 M_4 \right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup |x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{ x \in l_{n_0}^{\infty} : M_3 \le x(n) \le M_4, \, n \ge n_0 \right\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \dfrac{1}{P_1(n+\tau_1)} \Big\{ \alpha - x(n+\tau_1) - P_2(n+\tau_1) x(n+\tau_1+\tau_2) \\ + \displaystyle\sum_{s=n+\tau_1}^{\infty} \left[Q_1(s)\, x(s-\sigma_1) - Q_2(s)\, x(s+\sigma_2) \right] \Big\}, n \ge n_1, \\ (Sx)(n_1), \qquad n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.8) and (2.9) respectively, it follows that

$$\begin{aligned} (Sx)(n) &\le \frac{1}{P_1(n+\tau_1)} \left(\alpha + \sum_{s=n+\tau_1}^{\infty} Q_1(s) x(s-\sigma_1) \right) \\ &\le \frac{1}{p_1} \left(\alpha + M_4 \sum_{s=n}^{\infty} Q_1(s) \right) \\ &= \frac{1}{p_1} \left(\alpha + M_4 \left(\frac{p_1 M_4 - \alpha}{M_4} \right) \right) \\ (Sx)(n) &\le M_4 \end{aligned}$$

We have

$$\begin{aligned} (Sx)(n) &\ge \frac{1}{P_1(n+\tau_1)} \Big(\alpha - x(n+\tau_1) - P_2(n+\tau_1) x(n+\tau_1+\tau_2) \\ &\quad - \sum_{s=n+\tau_1}^{\infty} Q_2(s)\, x(s+\sigma_2) \Big) \\ &\ge \frac{1}{P_1(n+\tau_1)} \left(\alpha - M_4 - p_2 M_4 - M_4 \sum_{s=n}^{\infty} Q_2(s) \right) \\ &\ge \frac{1}{p_{1_0}} \left(\alpha - (1+p_2) M_4 - M_4 \left(\frac{\alpha - p_{1_0} M_3 - (1+p_2) M_4}{M_4} \right) \right) \\ (Sx)(n) &\ge M_3 \end{aligned}$$

Hence

$$M_3 \le (Sx)(n) \le M_4 \text{ for } n \ge n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\begin{aligned} &\left| (Sx_1)(n) - (Sx_2)(n) \right| \\ &\le \frac{1}{P_1(n+\tau_1)} \Big(\left| x_1(n+\tau_1) - x_2(n+\tau_1) \right| \\ &\quad + P_2(n+\tau_1) \left| x_1(n+\tau_1+\tau_2) - x_2(n+\tau_1+\tau_2) \right| \\ &\quad + \sum_{s=n+\tau_1}^{\infty} Q_1(s) \left| x_1(s-\sigma_1) - x_2(s-\sigma_1) \right| \\ &\quad + \sum_{s=n+\tau_1}^{\infty} Q_2(s) \left| x_1(s+\sigma_2) - x_2(s+\sigma_2) \right| \Big) \\ &\le \frac{1}{p_1} \Big(\|x_1 - x_2\| + p_2 \|x_1 - x_2\| + \sum_{s=n}^{\infty} Q_1(s) \|x_1 - x_2\| \\ &\quad + \sum_{s=n}^{\infty} Q_2(s) \|x_1 - x_2\| \Big) \\ &= \frac{1}{p_1} \left(1 + p_2 + \sum_{s=n}^{\infty} Q_1(s) + \sum_{s=n}^{\infty} Q_2(s) \right) \|x_1 - x_2\| \\ &= \frac{1}{p_1} \left(1 + p_2 + \frac{p_1 M_4 - \alpha}{M_4} + \frac{\alpha - p_{1_0} M_3 - (1+p_2) M_4}{M_4} \right) \\ &\quad \|x_1 - x_2\| \\ &= \frac{1}{p_1} \left(\frac{p_1 M_4 - p_{1_0} M_3}{M_4} \right) \|x_1 - x_2\| \\ &= \lambda_3 \|x_1 - x_2\| \end{aligned}$$

where $\lambda_3 = 1 - \dfrac{p_{1_0} M_3}{p_1 M_4}$. This implies that

$$\left\| (Sx_1)(n) - (Sx_2)(n) \right\| \le \lambda_3 \|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left| (Sx_1)(n) - (Sx_2)(n) \right| \le p(n) \left| x_1(n-\tau_1) - x_2(n-\tau_1) \right| \le \lambda_3 \|x_1 - x_2\|.$$

Since $0 < \lambda_3 < 1$,we conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.4. Assume that $1 < p_1 \le P_1(n) < p_{1_0} < \infty$, $1 - p_1 < p_2 \le P_2(n) \le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. In view of (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{(p_1+p_2)N_4-\alpha}{N_4}, n \ge n_1, \tag{2.10}$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{\alpha - p_{1_0}N_3 - N_4}{N_4}, n \ge n_1. \tag{2.11}$$

where N_3 and N_4 are positive constants such that $p_{1_0}N_3 + N_4 < (p_1+p_2)N_4$ and $\alpha \in \left(p_{1_0}N_3 + N_4, (p_1+p_2)N_4\right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{x \in l_{n_0}^{\infty} : N_3 \le x(n) \le N_4,\ n \ge n_0\right\}.$$

Define a mapping $S:\Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \frac{1}{P_1(n+\tau_1)}\{\alpha - x(n+\tau_1) - P_2(n+\tau_1)x(n+\tau_1+\tau_2) \\ + \sum_{s=n+\tau_1}^{\infty}\left[Q_1(s)\,x(s-\sigma_1) - Q_2(s)\,x(s+\sigma_2)\right]\}, & n \ge n_1, \\ (Sx)(n_1), & n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.10) and (2.11) respectively, it follows that

$$\begin{aligned}
(Sx)(n) &\le \frac{1}{P_1(n+\tau_1)}\Big(\alpha - P_2(n+\tau_1)x(n+\tau_1+\tau_2) \\
&\quad + \sum_{s=n+\tau_1}^{\infty} Q_1(s)\,x(s-\sigma_1)\Big) \\
&\le \frac{1}{p_1}\left(\alpha - p_2N_4 + N_4\sum_{s=n}^{\infty}Q_1(s)\right) \\
&= \frac{1}{p_1}\left(\alpha - p_2N_4 + N_4\left(\frac{(p_1+p_2)N_4-\alpha}{N_4}\right)\right)
\end{aligned}$$

$$(Sx)(n) \le N_4$$

Furthermore

$$\begin{aligned}
(Sx)(n) &\ge \frac{1}{P_1(n+\tau_1)}\Big(\alpha - x(n+\tau_1) \\
&\quad - \sum_{s=n+\tau_1}^{\infty} Q_2(s)\,x(s+\sigma_2)\Big) \\
&\ge \frac{1}{p_{1_0}}\left(\alpha - N_4 - N_4\sum_{s=n}^{\infty}Q_2(s)\right) \\
&= \frac{1}{p_{1_0}}\left(\alpha - N_4 - N_4\left(\frac{\alpha - p_{1_0}N_3 - N_4}{N_4}\right)\right)
\end{aligned}$$

$$(Sx)(n) \ge N_3$$

Hence

$$N_3 \le (Sx)(n) \le N_4 \text{ for } n \ge n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\begin{aligned}
&\left|(Sx_1)(n) - (Sx_2)(n)\right| \\
&\le \frac{1}{p_1}\Big(\|x_1-x_2\| - p_2\|x_1-x_2\| + \sum_{s=n}^{\infty}Q_1(s)\|x_1-x_2\| \\
&\quad + \sum_{s=n}^{\infty}Q_2(s)\|x_1-x_2\|\Big) \\
&= \frac{1}{p_1}\left(1 - p_2 + \sum_{s=n}^{\infty}Q_1(s) + \sum_{s=n}^{\infty}Q_2(s)\right)\|x_1-x_2\| \\
&= \frac{1}{p_1}\left(1 - p_2 + \frac{(p_1+p_2)N_4-\alpha}{N_4} + \frac{\alpha - p_{1_0}N_3 - N_4}{N_4}\right) \\
&\quad \|x_1-x_2\| \\
&= \frac{1}{p_1}\left(\frac{p_1N_4 - p_{1_0}N_3}{N_4}\right)\|x_1-x_2\| \\
&= \lambda_4\|x_1-x_2\|
\end{aligned}$$

where $\lambda_4 = 1 - \frac{p_{1_0}N_3}{p_1N_4}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \le \lambda_4\|x_1-x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \le p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \le \lambda_4\|x_1-x_2\|.$$

Since $0 < \lambda_4 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.5. Assume that $-1 < p_1 \le P_1(n) \le 0$, $0 \le P_2(n) \le p_2 < 1 + p_1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. From (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{(1+p_1)M_6 - \alpha}{M_6}, n \ge n_1, \qquad (2.12)$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{\alpha - p_2 M_6 - M_5}{M_6}, n \ge n_1. \qquad (2.13)$$

where M_5 and M_6 are positive constants such that $M_5 + p_2 M_6 < (1+p_1)M_6$ and $\alpha \in \left(M_5 + p_2 M_6, (1+p_1)M_6\right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{x \in l_{n_0}^{\infty} : M_5 \le x(n) \le M_6,\, n \ge n_0\right\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ +\sum_{s=n}^{\infty}\left[Q_1(s)\, x(s-\sigma_1) - Q_2(s)\, x(s+\sigma_2)\right], & n \ge n_1, \\ (Sx)(n_1), & n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.12) and (2.13) respectively, it follows that

$$(Sx)(n) \le \alpha - P_1(n)x(n-\tau_1) + \sum_{s=n}^{\infty} Q_1(s)\, x(s-\sigma_1)$$

$$\le \alpha - p_1 M_6 + M_6 \sum_{s=n}^{\infty} Q_1(s)$$

$$= \alpha - p_1 M_6 + M_6 \left(\frac{(1+p_1)M_6 - \alpha}{M_6}\right)$$

$$(Sx)(n) \le M_6$$

Also we have

$$(Sx)(n) \ge \alpha - P_2(n)x(n+\tau_2) - \sum_{s=n}^{\infty} Q_2(s)\, x(s+\sigma_2)$$

$$\ge \alpha - p_2 M_6 - M_6 \sum_{s=n}^{\infty} Q_2(s)$$

$$= \alpha - p_2 M_6 - M_6 \left(\frac{\alpha - p_2 M_6 - M_5}{M_6}\right)$$

$$(Sx)(n) \ge M_5$$

Hence

$$M_5 \le (Sx)(n) \le M_6 \text{ for } n \ge n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, it remains to show that S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\left|(Sx_1)(n) - (Sx_2)(n)\right|$$

$$\le -p_1\|x_1 - x_2\| + p_2\|x_1 - x_2\| + \sum_{s=n}^{\infty} Q_1(s)\|x_1 - x_2\|$$

$$+\sum_{s=n}^{\infty} Q_2(s)\|x_1 - x_2\|$$

$$= \left(-p_1 + p_2 + \sum_{s=n}^{\infty} Q_1(s) + \sum_{s=n}^{\infty} Q_2(s)\right)\|x_1 - x_2\|$$

$$= \left(-p_1 + p_2 + \frac{(1+p_1)M_6 - \alpha}{M_6} + \frac{\alpha - p_2 M_6 - M_5}{M_6}\right)$$

$$\|x_1 - x_2\|$$

$$= \frac{M_6 - M_5}{M_6}\left(\|x_1 - x_2\|\right)$$

$$= \lambda_5 \|x_1 - x_2\|$$

where $\lambda_5 = 1 - \frac{M_5}{M_6}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \le \lambda_5 \|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \le p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \le \lambda_5 \|x_1 - x_2\|.$$

Since $0 < \lambda_5 < 1$, we conclude that S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.6. Assume that $-1 < p_1 \le P_1(n) \le 0$, $-1 - p_1 < p_2 \le P_2(n) \le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. From (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.2) such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{(1+p_1+p_2)N_6 - \alpha}{N_6},\ n \ge n_1, \qquad (2.14)$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{\alpha - N_5}{N_6},\ n \ge n_1 \qquad (2.15)$$

where N_5 and N_6 are positive constants such that

$N_5 < (1+p_1+p_2)N_6$ and $\alpha \in \left(N_5, (1+p_1+p_2)N_6\right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{x \in l_{n_0}^{\infty} : N_5 \le x(n) \le N_6,\ n \ge n_0\right\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ +\sum_{s=n}^{\infty}\left[Q_1(s)\,x(s-\sigma_1) - Q_2(s)\,x(s+\sigma_2)\right], n \ge n_1, \\ (Sx)(n_1), \qquad n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.14) and (2.15) respectively, it follows that

$$\begin{aligned}(Sx)(n) &\le \alpha - P_1(n)x(n-\tau_1) - P_2(n)x(n+\tau_2) \\ &\quad +\sum_{s=n}^{\infty} Q_1(s)\,x(s-\sigma_1) \\ &\le \alpha - p_1N_6 - p_2N_6 + N_6\sum_{s=n}^{\infty} Q_1(s) \\ &= \alpha - p_1N_6 - p_2N_6 \\ &\quad +N_6\left(\frac{(1+p_1+p_2)N_6 - \alpha}{N_6}\right) \\ (Sx)(n) &\le N_6\end{aligned}$$

Furthermore

$$\begin{aligned}(Sx)(n) &\ge \alpha - \sum_{s=n}^{\infty} Q_2(s)\,x(s+\sigma_2) \\ &\ge \alpha - N_6\sum_{s=n}^{\infty} Q_2(s) \\ &= \alpha - N_6\left(\frac{\alpha - N_5}{N_6}\right) \\ (Sx)(n) &\ge N_5\end{aligned}$$

Hence

$$N_5 \le (Sx)(n) \le N_6 \text{ for } n \ge n_1.$$

Thus we have proved that $(Sx)(n) \in \Omega$ for any $x \in \Omega$. This means that $S\Omega \subset \Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\begin{aligned}&\left|(Sx_1)(n) - (Sx_2)(n)\right| \\ &\le -p_1\|x_1 - x_2\| - p_2\|x_1 - x_2\| + \sum_{s=n}^{\infty} Q_1(s)\|x_1 - x_2\| \\ &\quad +\sum_{s=n}^{\infty} Q_2(s)\|x_1 - x_2\| \\ &= \left(-p_1 - p_2 + \sum_{s=n}^{\infty} Q_1(s) + \sum_{s=n}^{\infty} Q_2(s)\right)\|x_1 - x_2\| \\ &= \left(-p_1 - p_2 + \frac{(1+p_1+p_2)N_6 - \alpha}{N_6} + \frac{\alpha - N_5}{N_6}\right) \\ &\quad \|x_1 - x_2\| \\ &= \frac{N_6 - N_5}{N_6}\left(\|x_1 - x_2\|\right) \\ &= \lambda_6\|x_1 - x_2\|\end{aligned}$$

where $\lambda_6 = 1 - \frac{N_5}{N_6}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \le \lambda_6\|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \le p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \le \lambda_6\|x_1 - x_2\|.$$

Since $0 < \lambda_6 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.7. Assume that $-\infty < p_{1_0} \le P_1(n) < p_1 < -1$, $0 \le P_2(n) \le p_2 < -p_1 - 1$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. In view of (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty} Q_1(s) \le \frac{p_{1_0}M_7 + \alpha}{M_8},\ n \ge n_1 \qquad (2.16)$$

$$\sum_{s=n}^{\infty} Q_2(s) \le \frac{(-p_1 - 1 - p_2)M_8 - \alpha}{M_8},\ n \ge n_1 \qquad (2.17)$$

where M_7 and M_8 are positive constants such that

$-p_{1_0}M_7 < (-p_1 - 1 - p_2)M_8$ and

$\alpha \in \left(-p_{1_0}M_7, (-p_1 - 1 - p_2)M_8\right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{ x \in l_{n_0}^{\infty} : M_7 \le x(n) \le M_8,\ n \ge n_0 \right\}.$$

Define a mapping $S : \Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n) = \begin{cases} \dfrac{-1}{P_1(n+\tau_1)}\{\alpha + x(n+\tau_1) + P_2(n+\tau_1)x(n+\tau_1+\tau_2) \\ -\sum\limits_{s=n+\tau_1}^{\infty}\left[Q_1(s)\,x(s-\sigma_1) - Q_2(s)\,x(s+\sigma_2)\right]\},\ n \ge n_1, \\ (Sx)(n_1), \qquad\qquad n_0 \le n \le n_1. \end{cases}$$

Obviously Sx is continuous. For $n \ge n_1$ and $x \in \Omega$, from (2.17) and (2.16) respectively, it follows that

$$\begin{aligned} (Sx)(n) &\le \frac{-1}{P_1(n+\tau_1)}\Big(\alpha + x(n+\tau_1) \\ &\quad + P_2(n+\tau_1)x(n+\tau_1+\tau_2) \\ &\quad + \sum_{s=n+\tau_1}^{\infty} Q_1(s)\,x(s-\sigma_1)\Big) \\ &\le \frac{-1}{p_1}\left(\alpha + M_8 + p_2M_8 + M_8\sum_{s=n}^{\infty}Q_2(s)\right) \\ &= \frac{-1}{p_1}(\alpha + M_8 + p_2M_8 \\ &\quad + M_8\left(\frac{(-p_1-1-p_2)M_8-\alpha}{M_8}\right)) \\ (Sx)(n) &\le M_8 \end{aligned}$$

Also

$$\begin{aligned} (Sx)(n) &\ge \frac{-1}{P_1(n+\tau_1)}\left(\alpha - \sum_{s=n+\tau_1}^{\infty} Q_1(s)\,x(s-\sigma_1)\right) \\ &\ge \frac{-1}{p_{1_0}}\left(\alpha - M_8\sum_{s=n}^{\infty}Q_1(s)\right) \\ &= \frac{-1}{p_{1_0}}\left(\alpha - M_8\left(\frac{p_{1_0}M_7+\alpha}{M_8}\right)\right) \\ (Sx)(n) &\ge M_7 \end{aligned}$$

Hence

$$M_7 \le (Sx)(n) \le M_8 \text{ for } n \ge n_1.$$

Hence $(Sx)(n) \in \Omega$ for any $x \in \Omega$.

This means that $S\Omega \subset \Omega$. In order to apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus if $x_1, x_2 \in \Omega$ and $n \ge n_1$,

$$\begin{aligned} &\left|(Sx_1)(n) - (Sx_2)(n)\right| \\ &\le \frac{-1}{p_1}\left(1 + p_2 + \sum_{s=n}^{\infty}Q_1(s) + \sum_{s=n}^{\infty}Q_2(s)\right)\|x_1 - x_2\| \\ &= \frac{-1}{p_1}\left(1 + p_2 + \frac{p_{1_0}M_7+\alpha}{M_8} + \frac{(-p_1-1-p_2)M_8-\alpha}{M_8}\right) \\ &\quad \|x_1 - x_2\| \\ &= \frac{-1}{p_1}\left(\frac{p_{1_0}M_7 - p_1M_8}{M_8}\right)\|x_1 - x_2\| \\ &= \lambda_7\|x_1 - x_2\| \end{aligned}$$

where $\lambda_7 = 1 - \dfrac{p_{1_0}M_7}{p_1M_8}$. This implies that

$$\left\|(Sx_1)(n) - (Sx_2)(n)\right\| \le \lambda_7\|x_1 - x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \ge n_1$ we have

$$\left|(Sx_1)(n) - (Sx_2)(n)\right| \le p(n)\left|x_1(n-\tau_1) - x_2(n-\tau_1)\right| \le \lambda_7\|x_1 - x_2\|.$$

Since $0 < \lambda_7 < 1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

Theorem 2.8. Assume that $-\infty < p_{1_0} \le P_1(n) < p_1 < -1$, $p_1 + 1 < p_2 \le P_2(n) \le 0$ and (2.1) hold, then (1.1) has a bounded non-oscillatory solution.

Proof. In view of (2.1), we can choose $n_1 \ge n_0$ sufficiently large satisfying (2.7) such that

$$\sum_{s=n}^{\infty}Q_1(s) \le \frac{p_{1_0}N_7 + p_2N_8 + \alpha}{N_8}, n \ge n_1, \qquad (2.18)$$

$$\sum_{s=n}^{\infty}Q_2(s) \le \frac{(-p_1-1)N_8 - \alpha}{N_8}, \quad n \ge n_1. \qquad (2.19)$$

where N_7 and N_8 are positive constants such that $-p_{1_0}N_7 - p_2N_8 < (-p_1-1)N_8$ and $\alpha \in \left(-p_{1_0}N_7 - p_2N_8, (-p_1-1)N_8\right)$.

Let $l_{n_0}^{\infty}$ be the set of all real sequence with the norm $\|x\| = \sup|x(n)| < \infty$. Then $l_{n_0}^{\infty}$ is a Banach space. We define a closed, bounded and convex subset Ω of $l_{n_0}^{\infty}$ as follows

$$\Omega = \left\{ x \in l_{n_0}^{\infty} : N_7 \le x(n) \le N_8,\ n \ge n_0 \right\}.$$

Define a mapping $S:\Omega \to l_{n_0}^{\infty}$ as follows

$$(Sx)(n)=\begin{cases}\dfrac{-1}{P_1(n+\tau_1)}\{\alpha+x(n+\tau_1)+P_2(n+\tau_1)x(n+\tau_1+\tau_2)\\ -\sum\limits_{s=n+\tau_1}^{\infty}\left[Q_1(s)\,x(s-\sigma_1)-Q_2(s)\,x(s+\sigma_2)\right]\}, n\geq n_1,\\ (Sx)(n_1), \qquad n_0\leq n\leq n_1.\end{cases}$$

Obviously Sx is continuous. For $n \geq n_1$ and $x \in \Omega$, from (2.19) and (2.18) respectively, it follows that

$$\begin{aligned}(Sx)(n) &\leq \frac{-1}{P_1(n+\tau_1)}\Big(\alpha+x(n+\tau_1)\\ &\quad+\sum_{s=n+\tau_1}^{\infty}Q_2(s)\,x(s+\sigma_2)\Big)\\ &\leq \frac{-1}{p_1}\left(\alpha+N_8+N_8\sum_{s=n}^{\infty}Q_2(s)\right)\\ &= \frac{-1}{p_1}\left(\alpha+N_8+N_8\left(\frac{(-p_1-1)N_8-\alpha}{N_8}\right)\right)\end{aligned}$$

$$(Sx)(n)\leq N_8$$

Furthermore

$$\begin{aligned}(Sx)(n) &\geq \frac{-1}{P_1(n+\tau_1)}\Big(\alpha+P_2(n+\tau_1)x(n+\tau_1+\tau_2)\\ &\quad-\sum_{s=n+\tau_1}^{\infty}Q_1(s)\,x(s-\sigma_1)\Big)\\ &\geq \frac{-1}{p_{1_0}}\left(\alpha+p_2N_8-N_8\sum_{s=n}^{\infty}Q_1(s)\right)\\ &= \frac{-1}{p_{1_0}}\left(\alpha+p_2N_8-N_8\left(\frac{p_{1_0}N_7+p_2N_8+\alpha}{N_8}\right)\right)\end{aligned}$$

$$(Sx)(n)\geq N_7$$

Hence

$$N_7 \leq (Sx)(n) \leq N_8 \text{ for } n \geq n_1.$$

Thus we have proved that $(Sx)(n)\in\Omega$ for any $x\in\Omega$. This means that $S\Omega\subset\Omega$. To apply contraction mapping principle, we shall show S is a contraction mapping on Ω. Thus $x_1, x_2 \in \Omega$ and $n \geq n_1$,

$$\begin{aligned}&\left|(Sx_1)(n)-(Sx_2)(n)\right|\\ &\leq \frac{-1}{p_1}\Big(\|x_1-x_2\|-p_2\|x_1-x_2\|+\sum_{s=n}^{\infty}Q_1(s)\|x_1-x_2\|\\ &\quad+\sum_{s=n}^{\infty}Q_2(s)\|x_1-x_2\|\Big)\\ &=\frac{-1}{p_1}\left(1-p_2+\sum_{s=n}^{\infty}Q_1(s)+\sum_{s=n}^{\infty}Q_2(s)\right)\|x_1-x_2\|\\ &=\frac{-1}{p_1}\left(1-p_2+\frac{p_{1_0}N_7+p_2N_8+\alpha}{N_8}+\frac{(-p_1-1)N_8-\alpha}{N_8}\right)\\ &\quad\|x_1-x_2\|\\ &=\frac{-1}{p_1}\left(\frac{p_{1_0}N_7-p_1N_8}{N_8}\right)\|x_1-x_2\|\\ &=\lambda_8\|x_1-x_2\|\end{aligned}$$

where $\lambda_8 = 1-\dfrac{p_{1_0}N_7}{p_1N_8}$. This implies that

$$\left\|(Sx_1)(n)-(Sx_2)(n)\right\|\leq\lambda_8\|x_1-x_2\|.$$

Thus we have proved that S is a contraction mapping on Ω. In fact $x_1, x_2 \in \Omega$ and $n \geq n_1$ we have

$$\left|(Sx_1)(n)-(Sx_2)(n)\right|\leq p(n)\left|x_1(n-\tau_1)-x_2(n-\tau_1)\right|\leq\lambda_8\|x_1-x_2\|.$$

Since $0<\lambda_8<1$, S is a contraction mapping on Ω. Thus S has a unique fixed point which is a positive and bounded solution of (1.1). This completes the proof.

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