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# Score Sequences of Tournaments with Repeated Values

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## ABSTRACT

In this paper we define score sequences with repetitions. The strong score sequences with a single repetition of length four are characterized and the total number of strong score sequences with a single repetition is obtained. We characterize the self-converse strong score sequences with a single repetition of length three and five and report the number of self-converse strong score sequences with a single repetition. In last we conjecture a result to find the number of self-converse strong score sequences with a single repetition of order n.

**Keywords:** Tournament, Score, Score Sequence.

## I. INTRODUCTION

A sequence of non-decreasing integers $(s_1,s_2,\ldots,s_n)$ is said to be realizable by a tournament T if there exists a tournament T with vertex set V(T) = $\{v_1, v_2,\ldots, v_n\}$ such that $s_i = s(v_i)$ for i = 1,2,…,n. Such a T is known as realization of S. First time Chen et al. [1] gave the concept of degree sequences with single repetitions for graphs in 1995. A score sequence S = $(s_1, s_2,\ldots, s_n)$ is said to contain a single repetition p of length q if there are exactly q equal entries $s_i = s_i+1 = \cdots = s_i+q-1 = p$, and all other entries of the sequence are distinct. Landau [3] characterized the score sequences:

## II. METHODS AND MATERIAL

**Theorem 1.[3]** A non-decreasing sequence S = $(s_1, s_2,\ldots, s_n)$ of non-negative integers is a score sequence of a tournament if and only if, for $1 \le k \le n$

$$\sum_{i=1}^{k} s_i \ge \frac{k(k-1)}{2}$$ , with equality for k = n.

**Corollary 2.[3]** A non-decreasing sequence S = $(s_1,s_2,\ldots,s_n)$ of non-negative integers is a strong score sequence of a tournament if and only if, for $1 \le k \le n-1$,

$$\sum_{i=1}^{k} s_i \ge \frac{k(k-1)}{2} \text{ and } \sum_{i=1}^{n} s_i = \frac{n(n-1)}{2}. \quad (1)$$

Beineke and Eggleton (unpublished) noted in the 1970's that one only needs check equation (1) for those values of k for which $s_k < s_k+1$ as reported in [4].

In 1979, Eplett [2] characterized the self-converse score sequences.

**Theorem 3.[2]** A score sequence S = $(s_1,s_2,\ldots,s_n)$ is self-converse if and only if S′= S, where S′ is the converse of S.

## III. RESULTS AND DISCUSSION

The following result shows the similarity between S′and S.

**Theorem 4.** S = $(s_1,s_2,\ldots,s_n)$ is a strong score sequence with a single repetition p of length q, if and only if the converse score sequence S′ = $(s_1',s_2',\ldots, s_n')$ is a strong score sequence with a single repetition n–1–p of the same length q, where

$$s_i' = n-1-s_i, \; i = 1,2,\ldots, n.$$

**Proof.** We know that S is strong if and only if S′ is strong. As S has a single repetition p of length q, then S′ will have a single repetition n–1–p of the same length q. Now we report the results which characterizes strong score sequences with single repetition of length four.

**Theorem 5(a).** For n even and $6j \le n < 6(j+1)$, j = 1,2,… there are 2j strong score sequences $S_k$ and $S_k'$ of order n, having a single repetition of length four, where

$$S_k = \left(1,2,\ldots,\frac{n+2(k-2)}{2}, 4\,x\,\frac{n+2(k-1)}{2}, \frac{n+2k}{2},\ldots,n-2\right) \backslash \left(\frac{n+6k-4}{2}\right), \text{ for } k=1,2,\ldots,j.$$

**Proof.** As n is even, let n=2m, now $3j \le m < 3(j+1)$, j = 1,2, …, and

$S_k$ = (1,2, …, m+k–2, 4x(m+k–1), m+k,…,2m–2) \(m+3k–2), k = 1,2,…, j.

Now we show that $S_k$ is a strong score sequence. Consider

$$\sum_{i=1}^{2m} s_i = \{1+2+\ldots+(2m-2)\} + 3(m+k-1) - (m+3k-2)$$

$$= \frac{2m(2m-1)}{2}$$ , thus the equality holds for n = 2m.

Now for strongness we have to show that

$$\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}, \quad 1 \le l < 2m$$

Case 1. For $1 \le l \le m+k-1$

$$\sum_{i=1}^{l} s_i = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}$$, thus the result holds for $1 \le l \le m+k-1$.

Case 2. For $m+k \le l \le m+k+2$

As $s_{m+k} = s_{m+k+1} = s_{m+k+2}$, we only consider l = m+k+2

$$\sum_{i=1}^{m+k+2} s_i = \sum_{i=1}^{m+k-1} s_i + 3(m+k-1)$$

$$= \frac{(m+k-1)(m+k+6)}{2}$$

$$= \frac{(m+k+2)(m+k+1)}{2} + (m+k-4),$$

$m \ge 3, k \ge 1.$ (2)

**Subcase 2.1.** When m = 3 and k = 1, then m + k – 4 = 0, so we get $S_1$= (1,2,3,3,3,3) which is strong.

**Subcase 2.2.** In the remaining two cases either $m \ge 3$ and $k > 1$ or $m > 3$ and $k \ge 1$, *i.e.*, m+k–4 > 0, from equation (2), we get

$$\sum_{i=1}^{m+k+2} s_i > \frac{(m+k+2)(m+k+1)}{2}$$ , hence the result holds for l = m+k+2.

As the missing entry is m+3k–2, so consider the case.

**Case 3.** For $m+k+3 \le l \le m+3k$.

$$\sum_{i=1}^{l} s_i = \sum_{i=1}^{m+k+2} s_i + \sum_{i=m+k+3}^{l} s_i$$

$$= \frac{(m+k-1)(m+k+6)}{2} + (s_{m+k+3} + s_{m+k+4} + \ldots + s_l)$$

$$= \frac{l(l-1)}{2} + (3m+3k-2l)$$

**Subcase 3.1.** For m = 3k, we get $4k+3 \le l \le 6k$ and equation (3) becomes,

$$\sum_{i=1}^{l} s_i = \frac{l(l-1)}{2} + 2(6k-l)$$, taking l < 6k, we get

$$\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}$$, for m = 3k.

**Subcase 3.2.** For m > 3k and $m+k+3 \le l \le m+3k$, from equation (3).

$$\sum_{i=1}^{l} s_i \ge \frac{l(l-1)}{2} + 3m+3k-2(m+3k)$$ , as $l \le m+3k$

or $$\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}$$, as m > 3k, so inequalities hold in this subcase.

**Case 4.** For $m+3k+1 \le l < 2m$

$$\sum_{i=1}^{l} s_i = \{1+2+\ldots+(l-2)\} + 3(m+k-1) - (m+3k-2)$$

$$= \frac{l(l-1)}{2} + (2m-l)$$

$$> \frac{l(l-1)}{2}$$, as l < 2m.

Hence $S_k$ for k = 1,2, …, j are strong score sequences of order n with a single repetition $\frac{n+2(k-1)}{2}$ of length four. Also by theorem (3) $S'_k$ for k = 1,2, …j are the strong score sequences of same order as $S_k$ with a single repetition $\frac{n-2k}{2}$ of length four. Which proves the theorem.

**Theorem 5(b).** For n odd and $6j+3 \le n < 6j+9$, j=1,2,…, there are 2j strong score sequences $S_k$ and $S_k'$ of order n, having a single repetition of length four, where

$S_k = (1,2,\ldots, \frac{n+2k-3}{2}, 4 \text{ x } \frac{n+2k-1}{2}, \frac{n+2k+1}{2},\ldots, n-2) \backslash (\frac{n+6k-1}{2})$

Where k = 1,2,…, j

**Proof.** The proof is similar to that of theorem 5(a).

**Theorem 6.** If n ≥ 3 and is odd, then

$S = (1,2,\ldots, \frac{n-3}{2}, 3 \text{ x } \frac{n-1}{2}, \frac{n+1}{2},\ldots, n-2)$ is a unique self-converse strong score sequence of order n with a single repetition $\frac{n-1}{2}$ of length three.

Proof. As n is odd, let n = 2m+1, m ≥ 1, S becomes,

S = (1,2,…, m–1, 3xm, m+1,…, 2m–1)

First we show that S is a strong score sequence. Consider

$\sum_{i=1}^{2m+1} s_i = \frac{(2m+1)(2m)}{2}$, hence equality holds for n = 2m+1.

It remains to prove that

$\sum_{i=1}^{l} s_i > \frac{l(l-1)}{2}$, for $1 \le l \le 2m$

**Case 1.** For $1 \le l \le m$, we have

$\sum_{i=1}^{l} s_i = \frac{l(l+1)}{2} > \frac{l(l-1)}{2}$, so inequalities hold in this case.

**Case 2.** For $m+1 \le l \le m+2$, as $s_{m+1} = s_{m+2}$, we only consider l = m+2,

$$\sum_{i=1}^{m+2} s_i = \frac{m(m+5)}{2}$$

**Subcase 2.1** Let m = 1, now S = (1,1,1), which is strong.

**Subcase 2.2** Let m > 1, we have

$\sum_{i=1}^{m+2} s_i = \frac{m(m+5)}{2} > \frac{(m+2)(m+1)}{2}$, for $m > 1$ so inequalities hold for $l = m+2$.

**Case 3.** Let $m+3 \le l \le 2m$, we have

$$\sum_{i=1}^{l} s_i = \frac{m(m+1)}{2} + 2m + \{(m+1) + (m+2) + \ldots + (l-2)\}$$

$$= \frac{m(m+5)}{2} + \frac{(l-m-2)(l+m-1)}{2}$$

$$= \frac{l^2-3l+4m+2}{2}$$

$$\ge \frac{l^2-3l+2l+2}{2}, \text{ (as } l \le 2m)$$

$= \frac{l(l-1)}{2} + 1 > \frac{l(l-1)}{2}$, so inequalities hold for $m+3 \le l \le 2m$.

Thus S is a strong score sequence with a single repetition $\frac{n-1}{2}$ of length three.

As S′ = S, S is self-converse.

Lastly, for uniqueness, consider a self-converse strong score sequence of order 2m+1, with a single repetition of length three, let

$S = (s_1, s_2, \ldots, s_m, s_{m+1}, s_{m+2}, \ldots, s_{2m+1})$ (4)

Where $s_i = t+i-1, 1 \le i \le m$

$s_{m+2}=s_{m+1}=s_m$

$s_i = 2m - s_{2m+2-i}, m+3 \le i \le 2m+1.$

Since $2s_{m+1} = 2m$ which gives t = 1 and hence S=(1, 2,…, m, m, m,…,2m), which is strong. □

The next result deals with the strong score sequences of even order, which has a single repetition p of length three.

**Corollary 7.** There is no strong score sequence of even order n (n ≥ 4), which has a single repetition of length three.

**Proof.** In the previous theorem we proved that the repeated entry of length three is $p = \frac{n-1}{2}$, hence n is even.

Now we characterize self-converse strong score sequences, having a single repetition of one entry of length five. We omit the proof.

**Theorem 8.** If n ≥ 5 and n odd, then there are $\frac{n-3}{2}$ self-converse strong score sequences $S_k$ of order n, having a single repetition $\frac{n-1}{2}$ of length five, where

$S_k = \left(1, 2, \ldots, \frac{n-3}{2}, 5\times\frac{n-1}{2}, \frac{n+1}{2}, \ldots, n-2\right) \setminus \left(\frac{n-1}{2}-k, \frac{n-1}{2}+k\right)$ for $k = 1, 2, \ldots, \frac{n-3}{2}$. Let sq(n) denote the number of strong score sequences with a single repetition of length q. Table1 lists the values of sq(n) for q = 4, 5, 6, and n = 6, 7,..., 15. No result is known to find the values of sq(n) for given q and n.

**Table 1**

| n | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|
| s4(n) | 2 | 0 | 2 | 2 | 2 | 2 | 4 | 2 | 4 | 4 |
| s5(n) | 0 | 2 | 2 | 3 | 6 | 6 | 8 | 11 | 14 | 14 |
| s6(n) | 0 | 0 | 2 | 4 | 6 | 10 | 16 | 20 | 28 | 40 |

Let sc(n) denote the number of self-converse strong score sequences with a single repetition of order n. Table 2 lists the values of sc(n) for some values of n.

**Table 2**

| n | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|
| sc(n) | 1 | 2 | 4 | 8 | 16 | 32 | 64 |

We conjecture that the value of $sc(n)$ is given by

$$sc(n) = 2^{\frac{(n-3)}{2}}$$ , for $n \geq 3$ and $n$ is odd.

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