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The 2-Outer Independent Domination Number of a Tree

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ABSTRACT

In this paper we presented that for every nontrivial tree T of order n with l leaves we have γ_2^{oi}(T) ≤ (n+l) / 2, and we characterized the trees attaining this upper bound and also we exhibited and characterized the common minimal equitable and vertex minimal equitable dominating graph which are either connected or complete.
Keywords: Tree, Upper Bound, Domination, Complete Bipartite Graph, Leaf, Eccentricity, 2-Outer Independent Dominating Set.

I. INTRODUCTION

Let G=(V,E) be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v)=\{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v, denoted by d(v), is the cardinality of its neighborhood .By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a support vertex is strong (weak, respectively) if it is adjacent to atleast two leaves (exactly one leaf, respectively). By G^* we denote the graph obtained from G by removing all leaves. The path on n vertices we denote by P_n.

We say that a subset of V(G) is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by α(G) is the maximum cardinality of an independent subset of V(G). An independent subset of the set of vertices of G of maximum cardinality is called an α(G)-set.

A subset D ⊆ V(G) is a dominating set of G if every vertex of V(G) - D has a neighbor in D, while it is a 2-dominating set of G if every vertex of V(G) - D has atleast two neighbors in D. The domination (2-domination, respectively) number of a graph G, denoted by γ(G) (γ_2(G), respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of G.

Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated atleast k times for a fixed positive integer k. Multiple domination was introduced by Fink and Jacobson [3], and further studied for example in [1, 2, 4, 5, 8, 10]. For a comprehensive survey of domination in graphs, we refer to [6,7].

A subset D ⊆ V(G) is a 2-outer-independent dominating set, abbreviated 2OIDS, of G if every vertex of V(G)-D has atleast two neighbors in D, and the set V(G)-D is independent. The 2-outer-independent domination number of G, denoted by γ_2^{oi} (G), is the minimum cardinality of a 2-outer-independent dominating set of G. A 2-outer-independent dominating set of G of minimum cardinality is called a γ_2^{oi}(G)-set. The study of 2-outer-independent domination in graphs was initiated in [9].

Blidia, Chellali and Favaron [1] established the following upper bound on the 2- domination number of a tree. For every nontrivial tree T of order n with l leaf we have γ_2(T) ≤ (n+l)/2. They also characterized the extremal trees.

We prove the following upper bound on the 2-outer-independent domination number of a tree. For every nontrivial tree T of order n with l leaves we have γ_2^{oi}(G) ≤ (n+l)/2. We also characterize the trees attaining this upper bound.

II. METHODS AND MATERIAL

Preliminaries

1. 2-Outer Independent Dominating Set

A subset D ⊆ V(G) is 2-outer independent dominating set of G if every vertex of V(G) – D has atleast two neighbors in D and the set V(G) – D is independent. It is denoted by 2OIDS.

The minimum cardinality of 2OIDS of G is called 2-outer independent dominating number and is denoted by γ_2^{oi}(G).

A 2-outer-independent dominating set of G of minimum cardinality is called a γ_2^{oi}(G)-set.

2. Leaf

A vertex of degree one is called end point (or) leaf. A support vertex is a vertex adjacent to a leaf.

3. Eccentricity

For any two points u, v of a graph we define the distance between u and v by

$$d(u,v) = \left\{\begin{matrix} \textit{the length of the shortest } u-v \textit{ path if such a path exists} \\ \infty \qquad\qquad \textit{otherwise} \end{matrix}\right\}$$

Let v be a point in a connected graph. The eccentricity ece(v) of v is defined by ece(v) = max {d(u, v) | u ∈ V(G)}.

III. RESULTS AND DISCUSSION

We begin with the following straightforward observation.

3.1 Observation

Every leaf of a graph G is in every γ_2^{oi}(G)-set.

Proof :

V(G)= {1,2,3,4,5,6,7},
D= {3,4,5,6,7},
V(G)-D= {1,2)

By the definition of γ_2^{oi}(G)-set, we have every vertex of V(G)-D has atleast two neighbors in D, and the set V(G)-D is independent.

Here {3,4,6,7} are leaves in a tree .
Hence the set D contains all leaves.
Hence every leaf of a graph G is in every γ_2^{oi}(G)-set.

We have the following relation between the 2-outer-independent domination number of a graph without isolated vertices and the independence number of the graph obtained from it by removing all leaves.

3.2. Lemma

If G is a graph without isolated vertices, then γ_2^{oi}(G) = n - α(G*).

Proof

Given G is a graph without isolated vertices.
We have to prove that γ_2^{oi}(G) = n - α(G*).
Let D be any γ_2^{oi}(G)-set.

By observation 3.1, "Every leaf of a graph G is in every γ_2^{oi}(G)-set", we have all leaves belong to the set D.

∴ V(G) - D ⊆ V(G*).
The set V(G) - D is independent.
Thus α(G*) ≥ │V(G) – D│
= n - γ_2^{oi}(G). ---------(1)
∴ α(G*) ≥ n - γ_2^{oi}(G).

Conversely,

Let D* be any α(G*)-set.

Let us observe that in the graph G every vertex of D* has atleast two neighbors in the set V(G) – D*.
∴ γ_2^{oi}(G) ≤ │ V(G) – D*│= n - α(G*).
∴ α(G*) ≤ n - γ_2^{oi}(G). ---------(2)
From (1) and (2) we have,
γ_2^{oi}(G) = n - α(G*).
Hence the proof.

Now we get an upper bound on the 2-outer-independent domination of bipartite graphs without isolated vertices.

3.3 Lemma

For every bipartite graph G without isolated vertices of order n with l leaves we have γ_2^{oi}(G) ≤ (n+l) / 2 .

Proof

Given G is any bipartite graph without isolated vertices of order n with l leaves.

To prove that γ_2^{oi}(G) ≤ (n+l)/2 .
Observe that G^* is also bipartite graph.

Thus there is an independent subset of the set of its vertices which contains atleast half of them.

∴ $\alpha(G^*) \geq |V(G^*)| / 2 = (n-l) / 2$.

Using the lemma 3.2, "If G is a graph without isolated vertices, then $\gamma_2^{oi}(G) = n - \alpha(G^*)$".

∴ we get $\gamma_2^{oi}(G) = n - \alpha(G^*)$

$$\leq n - [(n-l) / 2]$$
$$= n- (n / 2) + (l / 2)$$
$$= (n+l) / 2.$$

∴ $\gamma_2^{oi}(G) \leq (n+l) / 2$.

Hence the proof.

By T_{max} we denote the family of trees whose 2-outer-independent domination number attains the upper bound from the previous lemma.

We have the following property of trees of the family T_{max}.

3.4 Lemma

Let T be a tree. We have $T \in T_{max}$ if and only if $\alpha(T^*) = n^* / 2$.

Proof

Given T be a tree and $T \in T_{max}$
To prove that $\alpha(T^*) = n^* / 2$.

If T is a tree of the family T_{max}, we have $\gamma_2^{oi}(T) = (n+l) / 2$.

∴ By lemma 3.2 , "If G is a graph without isolated vertices, then $\gamma_2^{oi}(G) = n - \alpha(G^*)$."

∴ We get $\alpha(T^*) = n - \gamma_2^{oi}(T)$

$$= n-\left(\frac{n+l}{2}\right)$$
$$= \frac{n-l}{2} = \frac{n^*}{2}$$

∴ $\alpha(T^*) = \frac{n^*}{2}$

Conversely, assume that $\alpha(T^*) = \frac{n^*}{2}$

$$= \frac{n-l}{2}$$

To prove that $T \in T_{max}$.

∴ By lemma 3.2 , "If G is a graph without isolated vertices,

then $\gamma_2^{oi}(G) = n - \alpha(G^*)$."

∴ $\alpha(T^*) = n - \gamma_2^{oi}(T)$

$$\frac{n-l}{2} = n - \gamma_2^{oi}(T)$$
$$n - \frac{n-l}{2} = \gamma_2^{oi}(T)$$

∴ $\frac{n+l}{2} = \gamma_2^{oi}(T)$

∴ $T \in T_{max}$.

Hence the proof.

We showed that if G is a bipartite graph without isolated vertices of order n with l leaves, then γ_2^{oi}(G) is bounded above by $\frac{n+l}{2}$.

We characterize all trees attaining this bound. For this purpose we introduce a family $\mathcal{T}$ of trees that can be obtained from P_2 by applying consecutively operations O_1 (or) O_2 defined below.

Operation O_1 : Add one new vertex and one edge joining this new vertex to a non-leaf vertex of a graph.

Operation O_2 : Add two new vertices, one edge joining them, and one edge joining one of them to a leaf of a graph.

Now we prove that for every tree of the family $\mathcal{T}$, the 2-outer independent domination number equals the number of leaves plus half of the remaining vertices.

3.5 Lemma

Any tree $T \in \mathcal{T}$ is in T_{max}.

Proof

We have $\gamma_2^{oi}(T) = \frac{n+l}{2} = \frac{2+2}{2} = 2$.

Thus $P_2 \in T_{max}$.

∴ The result is true for the starting tree.

It remains to show that performing the operations O_1 and O_2 keeps the property being in T_{max}.

Let T be a tree obtained for $T' \in \mathcal{T}$ by operation O_1.

We have $T^* = T'^*$.

If $T' \in T_{max}$, then by lemma3.4, " Let T be a tree.

We have $T \in T_{max}$ if and only if $\alpha(T^*) = n^* / 2$ ".

This implies that $T \in T_{max}$.

Now, let T be a tree obtained from $T' \in \mathcal{T}$ by operation O_2.

We have $n^* = n'^* + 2$.

Let us observe that

$\alpha(T^*) = \alpha[(T')^*]+1$.

If $T' \in T_{max}$, then by lemma 3.4 , "Let T be a tree.

We have $T \in T_{max}$ if and only if

$\alpha(T^*) = n^* / 2$ ".

This implies that $T \in T_{max}$.

Hence the proof.

Now we prove that if the 2-outer independent domination number of a tree equals the number of leaves plus half of the remaining vertices, then the tree belongs to the family $\mathcal{T}$.

3.6 Lemma

Any tree $T \in T_{max}$ is in $\mathcal{T}$.

Proof

We prove that the result by the induction on the number n of vertices of T.

If it has only two vertices, then $T = P_2 \in \mathcal{T}$.

Now assume that $n \geq 3$.

Assume that the result is true for every tree T' of order $n' < n$.

Assume that some support vertex of T, say x has degree atleast three.

Let y be a leaf adjacent to x.

Let $T' = T - y$.

We have $T'^* = T^*$.

By lemma 3.4 , " Let T be a tree. We have $T \in T_{max}$ if and only if

$\alpha(T^*) = n^* / 2$".

This implies that $T' \in T_{max}$.

By the inductive hypothesis we have $T' \in \mathcal{T}$.

The tree T can be obtained from T' by operation O_1.

Thus $T \in \mathcal{T}$.

Henceforth, we can assume that every support vertex of T has degree two.

We now root T at a vertex r of maximum eccentricity.

Let t be a leaf at maximum distance from r, v be a parent of t and u be the parent of v in the rooted tree.

By T_x let us denote the subtree induced by a vertex x and its descendants in the rooted tree T.

First assume that $d_T(u) \geq 3$.

Let x be a descendant of u other than v.

Since every support vertex of T has degree two, the vertex x is not a leaf.

Thus it is a support vertex.

Let $T' = T - T_v$.

Let us observe that $n'^* = n^* - 1$ and $\alpha(T'^*) = \alpha(T^*) - 1$.

Using lemma 3.4, "Let T be a tree. We have $T \in T_{max}$ if and only if $\alpha(T^*) = n^* / 2$".

∴ we get $\alpha(T'^*) = \alpha(T^*) - 1$

$$= \frac{n^*}{2} - 1 = \frac{n'^*}{2} - 1$$

$$= \frac{n^*}{2} - \frac{1}{2} < \frac{n'^*}{2} .$$

This is a contradiction as T'^* is bipartite graph.

Now assume that $d_T(u) = 2$.

Let $T' = T - T_v$.

Let us observe that $n'^* = n^* - 2$ and $\alpha(T'^*) = \alpha(T^*) - 1$.

Now we get $\alpha(T'^{*}) = \alpha(T^{*}) - 1 = \frac{n^{*}}{2} - 1 = \frac{n^{*}-2}{2} = \frac{n'^{*}}{2}$.

By lemma 3.4, " Let T be a tree. We have $T \in T_{max}$ if and only if $\alpha(T^{*}) = n^{*} / 2$".

This implies that $T' \in T_{max}$.
By the inductive hypothesis we have $T' \in \mathcal{T}$.

The tree T can be obtained from T' by operation O_2.

Thus $T \in \mathcal{T}$.

Hence the proof.

As a consequence of lemmas we get the final result, an upper bound on the 2-outer independent domination number of a tree together with the characterization of the extremal trees.

3.7 Theorem

If T is a nontrivial tree of order n with l leaves, then $\gamma_2^{oi}(T) \leq (n+l) / 2$ with equality iff $T \in \mathcal{T}$.

Proof

Given T is a nontrivial tree of order n with l leaves.

To prove that $\gamma_2^{oi}(T) \leq (n+l) / 2$.

Also, we prove that the equality holds when $T \in \mathcal{T}$.

By lemma 3.3 , "For every bipartite graph G without isolated vertices of order n with l leaves we have $\gamma_2^{oi}(G) \leq (n+l) / 2$ ".

$\therefore \gamma_2^{oi}(T) \leq (n+l) / 2$.

Next we assume that

$\gamma_2^{oi}(T) = (n+l) / 2$.

To prove that $T \in \mathcal{T}$.

By lemma 3.6 , "Any tree $T \in T_{max}$ is in $\mathcal{T}$".

By the definition of T_{max}, "The family of trees whose 2-outer-independent domination number attains the upper bound".

i.e., $\gamma_2^{oi}(T) = (n+l) / 2$.

Hence we have $T \in \mathcal{T}$.

Conversely, assume that $T \in \mathcal{T}$.

To prove that $\gamma_2^{oi}(T) = (n+l) / 2$.

By lemma 3.5 and 3.6, "Any tree $T \in T_{max}$ is in $\mathcal{T}$" and "Any tree $T \in \mathcal{T}$ is in T_{max}".

Hence we have $\gamma_2^{oi}(T) = (n+l) / 2$.

Hence the proof.

IV. CONCLUSION

In this paper, we presented that for every nontrivial tree T of order n with l leaves we have $\gamma_2^{oi}(T) \leq (n+l) / 2$, and we characterized the trees attaining this upper bound and also we exhibited and characterized the common minimal equitable and vertex minimal equitable dominating graph which are either connected or complete.

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