# Model for the Dynamical Study of a Three-Species Food-Chain System under Toxicant Stress 

Raveendra Babu. A , O. P. Misra, Chhatrapal Singh, Preety Kalra<br>School of Mathematics and Allied Sciences, Jiwaji University, Gwalior, India


#### Abstract

The modeling investigation in this paper discusses the system level effects of a toxicant on a three species food chain system and the state variables of the models are prey and predator densities, concentration of toxicant in the environment and the concentration of toxicant in the prey population. In the models, we have assumed that the presence of top predator reduces the predatory ability of the intermediate predator. The stability analysis of the models is carried out and the sufficient conditions for the existence and extinction of the populations under the stress of toxicant are obtained. Further, it is also found that the predation rate of the intermediate predator is a bifurcating parameter and Hopf-bifurcation occurs at some critical value of this parameter. Finally, numerical simulation is carried out to support the analytical results.


Keywords: Stability, Bifurcation, Lyapunov function

## I. INTRODUCTION

Species are regularly exposed to many natural and synthetic chemicals which are adversely affecting their growth rate directly or indirectly. The direct effects of toxicant on the species are alterations in their mortality and reproductive rates. The indirect effects are observed either through the food chain or through the reduction in the carrying capacity of the environment due to the degradation of the habitat. It is generally observed in nature that the toxicants decrease the growth rate of species and also their carrying capacity. The presence of toxicants in the environment affects not only the species but their resources also. These toxicants have very pronounced effects on the species if the availability of the resource is limited. There are many instances where the toxicants have been the main cause of extinction of many species and depletion of resources such as forestry, fertile crop and wild life.

Ecologists and mathematicians have often used food chain systems to describe the feeding relationships between species within ecosystems and there has been considerable interest in the predator-prey models, especially for systems of three species (Freedman and Waltman, 1977; Hastings and Powell, 1991; Gakkhar
and Naji, 2003; Gakkhar and Singh, 2006; Gakkhar et al., 2007; Naji and Balasim, 2007; Gomes et al., 2008; Wang and Pang, 2008; Zhao and Lv, 2009a; George et al., 2010; Naji et al., 2010; Mada et al., 2011; Zhao et al., 2011; Zhuang and Wen, 2011; Haque et al., 2013; Wang and Zhao, 2011; Kumari, 2013; Jana, et al., 2014). However, the ecological communities in nature are observed to exhibit very complex dynamical behaviors and three species continuous time models are reported to have more complicated patterns. In (Lv and Zhao, 2008) have proposed and examined the dynamic complexities of a three species food chain model and found different forms of complexities in their model. In (Sun and Loreau, 2009) proposed a three-species food chain model with dynamically variable adaptive traits in the intermediate consumer and from the stability analysis they have shown that the positive equilibrium is globally stable under specific conditions. However, recently in (Gomes et al., 2008) have considered the classical fishpond management for tilapia fish culture model and studied three levels consisting of young tilapia (prey), developed tilapia (predator) and tucunare fish (top-predator) in order to describe the dynamical behavior of a three-species food chain system. It may be noted here that these studies have not incorporated the
effects of toxicants on the survival or extinction of prey populations in the food chain systems.

The study of the effects of toxic substances on ecological communities is of great interest, both from environmental and conservational points of view. Species exposed to polluted environment become vulnerable to several stresses due to which their existence may be threatened in long run. In the experiment study of (Smith and Weis, 1997) the authors have observed that the fish from the polluted environment suffered significantly greater mortality in the presence of a predator, the blue crab Callinectes sapidus Rathbun, than fish from the unpolluted environment. In the study of (Jes et al., 2013) the authors evaluated that the during exposure to sublethal concentrations of LC (lambda-cyhalothrin) the predatorprey interactions between G. pulex and L. nigra were significantly altered. The relative frequency of successful predation by G. pulex on L. nigra decreased from nearly 100 percent in the control and the $<1 \mathrm{ng} L^{-1}$ treatments to approximately 50 percent in the $6.6 \mathrm{ng} L^{-1}$ treatment, and no predation was observed in the 62.1 ng $L^{-1}$ treatment during the 60 min observation period. These findings probably reflect an increased stress response of G. pulex to increasing concentrations of LC prompting behavioural hyperactivity that overrules the natural instinct of catching the prey. So, in order to use and regulate toxic substances wisely, we must asses the risk of the populations exposed to toxicants. Some investigators have studied the effects of toxicant on one and two interacting species systems using mathematical models (Hallam and Clark, 1982; Hallam et al, 1983; Hallam and Luna, 1984; Luna and Hallam, 1987; Freedman and Shukla, 1991; Misra and Saxena, 1991; Shukla and Dubey, 1996; Shukla, 2001; Hamilton, 2004; Misra and Sinha, 2007; Das et al, 2009; Sinha et al., 2010; Khare et al., 2011; Agarwal and Devi, 2011; Huang et al., 2013). Previously, some research have been done on tri-trophic food-chain systems including toxicant effects on the survival or extinction of species in the system (Hallam and Luna, 1984; Thomann et al., 1984; Misra and Babu.A, 2014).

In this paper therefore we have studied the dynamical behaviour of a three-species food-chain system under toxicant stress considering modified Smith model for
prey species (Hallam and Luna, 1984) and predatory interference by top predator using mathematical model.

## II. MATHEMATICAL MODEL

The model formulation has been carried out in the light of the research papers of (Lv and Zhao, 2008) and (Hallam and Luna, 1984). In the model, the underlying food chain system consists of a prey population, an intermediate predator population and a top predator population with Holling type-II functional responses. It is assumed in the model that the presence of the top predator reduces the predatory ability of the intermediate predator (Lv and Zhao, 2008). In the model, the growth equation for the prey population in the absence of predator is assumed to be governed by a modified Smith-type differential equation (Hallam and Luna, 1984). The state variables of the model are $x(t)$, the density of the prey population; $y(t)$, the density of the intermediate predator population; $z(t)$, the density of the top predator population; $C_{0}(t)$, the organism toxicant concentration in the prey population; and $C_{E}(t)$, the environmental toxicant concentration.
Taking these as state variables, we formulate the mathematical model using following system of non linear ordinary differential equations in order to study the effect of toxicant on a three-species food chain system:

## Model 1:(with toxicant)

$\frac{d x}{d t}=x\left(\frac{r\left(C_{0}\right) B\left(c+r_{0}-a\right)-a c x}{B\left(c+r_{0}-a\right)+a x}\right)-\left(\frac{A_{1} x}{B_{1}+x}\right) \frac{y}{1+z}$
$\frac{d y}{d t}=\beta_{1}\left(C_{0}\right)\left(\frac{A_{1} x}{B_{1}+x}\right) \frac{y}{1+z}-\left(\frac{A_{2} y}{B_{2}+y}\right) z-D_{1} y$
$\frac{d z}{d t}=\beta_{2}\left(C_{0}\right)\left(\frac{A_{2} y}{B_{2}+y}\right) z-D_{2} z$
$\frac{d C_{0}}{d t}=a_{1} C_{E}+\frac{d_{1}}{a_{1}} \theta \beta-\left(l_{1}+l_{2}\right) C_{0}$
$\frac{d C_{E}}{d t}=g_{0}+k_{1} l_{1} C_{0}^{x-k_{1}} a_{1} C_{E} x-k_{2} C_{E}$

The initial conditions are
$\mathrm{x}(0)=\mathrm{x}_{0}>0, \quad \mathrm{y}(0)=\mathrm{y}_{0}>0, \quad \mathrm{z}(0)=\mathrm{z}_{0}>0, \quad \mathrm{C}_{0}(0)=0$, $C_{E}(0)=C_{E}>0$.

Where, $B$ is the population carrying capacity; $r_{0}$ is the intrinsic growth rate of the population; $a$ is a measure of the population response to stress effects; $D_{1}$ and $D_{2}$ are the death rates of $y$ and $z$ respectively.

We assume that the toxicant concentration $\theta$ in the population is a constant; $\beta$ is the average rate of food intake per unit organismal mass; $l_{1}$ and $l_{2}$ are egestion and depuration rates respectively; $k_{1} l_{1} C_{0} x$ is the total toxicant ingested; $k_{1} a_{1} C_{E} x$ is the total toxicant uptake from the environment; $k_{2} C_{E}$ is the term which describes the loss due to detoxifying process such as hydrolysis, volatilization, etc.; The exogenous input to the body burden $C_{0}$ is assumed to be from the environment at a rate proportional to the environmental concentration: $a_{1} C_{E}$, where $a_{1}$ is the population rate of toxicant uptake per unit mass; $g_{0}$ represents exogenous input of toxicant into the environment; $d_{1}$ is a constant numerically less than or equal to the numerical value of $a_{1} ; c$ is the rate of replacement of mass in the population at saturation. $\beta_{i}\left(C_{0}\right)$ is the biomass conversion rate, $i=1,2$., where $\beta_{i}(0)=\beta_{i 0}$ and $\beta_{i}\left(C_{0}\right)<0$.

In the $\operatorname{model} A_{i} u /\left(B_{i}+u\right),(i=1,2 ; u=x$ and $y)$, account for the interactions between two different species, representing the Holling type-II functional response. This functional response is parameterized by the constants $A_{i}$ and $B_{i}(i=1,2)$, and we verify that $B_{i}$ is the value of the prey population level when the predation rate per unit prey is half their maximum value ( Lv and Zhao, 2008).

Exposure to toxicant may lead to changes in fecundity and mortality rates of a population. This stress can be modelled by assuming that the growth rate of the population is a function of the body burden $r\left(C_{0}\right)=r_{0}-H\left(C_{0}\right)$. Here $H$ is a non- decreasing function of $C_{0}$ with $H(0)=0$ and $r_{0}$ is the intrinsic growth rate of the population. $H\left(C_{0}\right)$ is a dose-response function, which is assumed to be linear and taken as $H\left(C_{0}\right)=r_{1} C_{0}$ (Hallam and Luna, 1984).

We can reduce the number of parameters in the above system by the following scaling transformations, even if, for our analytical and numerical tests, we will continue to use the original system:

$$
\begin{gathered}
x \rightarrow \frac{x}{B_{1}}, y \rightarrow \frac{A_{1} y}{T_{0}}, z \rightarrow \frac{A_{1} z}{A_{2} T_{0}}, \\
C_{O} \rightarrow \frac{r_{0} C_{O}}{k_{1}}, C_{E} \rightarrow \frac{r_{0} C_{E}}{a_{1}}, t \rightarrow D_{1} t .
\end{gathered}
$$

Thus, system (1)-(5) after re-scaling becomes as follows:
$\frac{d x}{d t}=x e_{1}\left(\frac{r_{1}\left(C_{0}\right) e_{2}-x}{e_{4}+x}\right)-\frac{u_{1} x y}{(1+x)\left(1+e_{5} z\right)}$
$\frac{d y}{d t}=\frac{\beta_{3}\left(C_{0}\right) e_{6} x y}{(1+x)\left(1+e_{5} z\right)}-\frac{u_{2} y z}{e_{8}+y}-y$
$\frac{d z}{d t}=\frac{\beta_{4}\left(C_{0}\right) e_{9} y z}{e_{8}+y}-u_{3} z$
$\frac{d C_{0}}{d t}=o_{2} C_{E}+o_{3}-u_{4} C_{0}$
$\frac{d C_{E}}{d t}=u_{0}+o_{4} C_{0} x-u_{5} C_{E} x-u_{6} C_{E}$
The initial conditions are
$x(0)=x_{0}>0, y(0)=y_{0}>0, z(0)=z_{0}>0, C_{0}(0)=0, C_{E}(0)=C_{E}>0$.
Here,

$$
\begin{gathered}
u_{0}=\frac{g_{0} a_{1}}{D_{1} r_{0}}, u_{1}=\frac{T_{0}}{B_{1}^{3} D_{1}}, u_{2}=\frac{T_{0}^{2}}{D_{1} A_{1}^{2}}, u_{3}=\frac{D_{2}}{D_{1}}, \\
u_{4}=\frac{\left(l_{1}+l_{2}\right)}{D_{1}}, u_{5}=\frac{a_{1} k_{1} B_{1}}{D_{1}}, u_{6}=\frac{k_{2}}{D_{1}}, o_{1}=\frac{\beta_{20} k_{1}}{r_{0} \beta_{22}}, \\
o_{2}=\frac{k_{1}}{D_{1}}, o_{3}=\frac{k_{1} d_{1} \theta \beta}{a_{1} r_{0} D_{1}}, o_{4}=\frac{a_{1} l_{1} B_{1}}{D_{1}}, e_{1}=\frac{1}{\left(a B_{1}\right)^{2}} \frac{1}{c D_{1}}, \\
e_{2}=\frac{r_{0} r_{1} T_{0}}{a c k_{1} B_{1}}, e_{3}=\frac{k_{1}}{r_{1}}, e_{4}=\frac{T_{0}}{a B_{1}}, e_{5}=\frac{A_{1}}{A_{2} T_{0}}, \\
T_{1}^{2} r_{0} \beta_{11} D_{1}, e_{7}=\frac{\beta_{10} k_{1}}{r_{0} \beta_{11}}, e_{8}=\frac{B_{2} T_{0}}{A_{1}}, e_{9}=\frac{r_{0} \beta_{11}}{k_{1} D_{1} A_{2}},
\end{gathered}
$$

All these parameters, of course, assume only positive values.

Now, if the effect of toxicant is not considered in the above Model 1, then we have the following Model 2 for three species food chain system:
Model 2:( without toxicant)
$\frac{d x}{d t}=\quad x\left(\frac{r_{0} B\left(c+r_{0}-a\right)-a c x}{B\left(c+r_{0}-a\right)+a x}\right)-\left(\frac{A_{1} x}{B_{1}+x}\right) \frac{y}{1+z}$
$\frac{d y}{d t}=\beta_{10}\left(\frac{A_{1} x}{B_{1}+x}\right) \frac{y}{1+z}-\left(\frac{A_{2} y}{B_{2}+y}\right) z-D_{1} y$
$\frac{d z}{d t}=\beta_{20}\left(\frac{A_{2} y}{B_{2}+y}\right) z-D_{2} z$
with the initial conditions as

$$
x(0)=x_{0}>0, y(0)=y_{0}>0, z(0)=z_{0}>0 .
$$

where, the state variables and parameters are the same as defined for the Model 1.
We can reduce the number of parameters in the above system, even if, for our analytical and numerical tests, we will continue to use the original system:

$$
o_{5}=e_{2} e_{3} ; o_{6}=e_{6} e_{7} ; o_{7}=e_{9} o_{1} .
$$

Rest of the parameters are the same as defined for the Model 1. All these parameters, of course, assume only positive values. Thus, system (11)-(13) after re-scaling becomes as follows:
$\frac{d x}{d t}=x e_{1}\left(\frac{o_{5}-x}{e_{4}+x}\right)-\frac{u_{1} x y}{(1+x)\left(1+e_{5} z\right)}$
$\frac{d y}{d t}=\frac{o_{6} x y}{(1+x)\left(1+e_{5} z\right)}-\frac{u_{2} y z}{e_{8}+y}-y$
$\frac{d z}{d t}=\frac{{ }_{7} y z}{e_{8}+y}-u_{3} z$
The initial conditions are

$$
x(0)=x_{0}>0, y(0)=y_{0}>0, z(0)=z_{0}>0 .
$$

## III. ANALYSIS OF MODEL 2

### 3.1 EQUILIBRIA OF MODEL 2

The Model 2 has following four non-negative equilibria. They are listed below:

$$
\begin{align*}
& 1 \cdot E_{0}^{*}=(0,0,0) \\
& 2 \cdot E_{1}^{*}=(\tilde{x}, 0,0) \\
& \quad \tilde{x}=r_{0} T_{0} / a c \tag{17}
\end{align*}
$$

where $T_{0}=B\left(c+r_{0}-a\right)$.
3. $E_{2}^{*}=(\hat{x}, \hat{y}, 0)$, where,

$$
\begin{equation*}
\hat{x}=D_{1} B_{1} /\left(\beta_{10} A_{1}-D_{1}\right)>0 \tag{18}
\end{equation*}
$$

provided $\beta_{10} A_{1}>D_{1}$,
$\hat{y}=\frac{B_{1} \beta_{10}}{\beta_{10} A_{1}-D_{1}}\left[\frac{r_{0} T_{0}\left(\beta_{10} A_{1}-D_{1}\right)-a c B_{1} D_{1}}{T_{0}\left(\beta_{10} A_{1}-D_{1}\right)+a B_{1} D_{1}}\right]>0$
Provided $r_{0} T_{0}\left(\beta_{10} A_{1}-D_{1}\right)>a c B_{1} D_{1}$
and $x^{*}$ is given by

$$
\begin{aligned}
& x^{* 2} a c \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right) \\
& -x^{*}\left[r_{0} T_{0} \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right)-a D_{1} D_{2} B_{2} \beta_{20}\right] \\
& +D_{1} D_{2} B_{2} T_{0} \beta_{20}=0 .
\end{aligned}
$$

$x^{*}, y^{*}$ and $z^{*}$ are positive provided $A_{2} \beta_{20}>D_{2} ; \omega_{2}>0$;

$$
\begin{equation*}
r_{0} T_{0} \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right)>a D_{1} D_{2} \beta_{20} B_{2} \tag{23}
\end{equation*}
$$

and
$\frac{\omega_{1}-\sqrt{\omega_{2}}}{2 a c \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right)}<x^{*}<\frac{\varpi_{1}+\sqrt{\omega_{2}}}{2 a c \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right)}$
where,

$$
\begin{aligned}
& \varpi_{1}=r_{0} T_{0} \beta_{10} \beta_{20}\left(A_{2} \beta_{20}-D_{2}\right)-a D_{1} D_{2} \beta_{20} B_{2} \\
& \varpi_{2}=\varpi_{1}^{2}-4 a c D_{1} D_{2} B_{2} T_{0} \beta_{10} \beta_{20}^{2}\left(A_{2} \beta_{20}-D_{2}\right) .
\end{aligned}
$$

### 3.2 DYNAMICAL BEHAVIOUR OF MODEL 2

The general variational matrix corresponding to the Model 2 is

$$
J(x, y, z)=\left[\begin{array}{ccc}
-m_{11} & -m_{12} & m_{13} \\
m_{21} & m_{22} & -m_{23} \\
0 & m_{32} & m_{33}
\end{array}\right]
$$

where,

$$
\begin{aligned}
& m_{11}=\frac{A_{1} B_{1} y}{\left(B_{1}+x\right)^{2}(1+z)}-\frac{r_{0} T_{0}^{2}-2 a c x T_{0}-a^{2} c x^{2}}{\left(T_{0}+a x\right)^{2}}, \\
& m_{12}=\frac{A_{1} x}{\left(B_{1}+x\right)(1+z)}, m_{13}=\frac{A_{1} x y}{\left(B_{1}+x\right)(1+z)^{2}},
\end{aligned}
$$

$m_{21}=\frac{A_{1} B_{1} \beta_{10} y}{\left(B_{1}+x\right)^{2}(1+z)}, m_{22}=\frac{A_{1} \beta_{10} x}{\left(B_{1}+x\right)(1+z)}-\frac{A_{2} B_{2} z}{\left(B_{2}+y\right)^{2}}-D_{1}$,
$m_{23}=\frac{A_{1} \beta_{10} x y}{\left(B_{1}+x\right)(1+z)^{2}}+\frac{A_{2} y}{B_{2}+y}, m_{32}=\frac{A_{2} B_{2} \beta_{20} z}{\left(B_{2}+y\right)^{2}}$,
$m_{33}=\frac{A_{2} \beta_{20} y}{B_{2}+y}-D_{2}$.

1. At $E_{0}^{*}$, the eigenvalues of the characteristic equation are $r_{0},-D_{1}$ and $-D_{2}$, which shows that $E_{0}^{*}$ is unstable. 2. At $E_{1}^{*}$, the eigen values of the characteristic equation are $-r_{0} c /\left(c+r_{0}\right), A_{1} \beta_{10} \tilde{x} /\left(B_{1}+\tilde{x}\right)-D_{1}$ and $-D_{2}$, which shows $E_{1}^{*}$ is locally asymptotically stable if

$$
\begin{equation*}
A_{1} \beta_{10} \tilde{x} /\left(B_{1}+\tilde{x}\right)<D_{1} \tag{25}
\end{equation*}
$$

holds good. We note from (25) that, $E_{1}^{*}$ of the system is locally asymptotically stable if the condition

$$
\begin{equation*}
r_{0} T_{0}\left(\beta_{10} A_{1}-D_{1}\right)<a c B_{1} D_{1} \tag{26}
\end{equation*}
$$

is being satisfied. The condition (26) is automatically satisfied if

$$
\begin{equation*}
A_{1} \beta_{10} \leq D_{1} \tag{27}
\end{equation*}
$$

and in this case $E_{2}^{*}$ does not exist. If $E_{2}^{*}$ exists then $E_{1}^{*}$ of the system is unstable.

Remark 1: From (27), it may be noted that if predation rate of intermediate consumer is less than or equal to the ratio of its death rate and its conversion efficiency then basal resource will only survive and intermediate consumer and top predator will go to extinction.
3. At $E_{2}^{*}$, one of the eigenvalue of the characteristic equation is $A_{2} \beta_{20} \hat{y} /\left(B_{2}+\hat{y}\right)-D_{2}$ and the other two eigenvalues are given by the roots of the following quadratic equation
$\lambda^{2}+\left[\frac{a \hat{x} T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a \hat{x}\right)^{2}}-\frac{A_{1} \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{2}}\right] \lambda+\left[\frac{A_{1}^{2} B_{1} \beta_{10} \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{3}}\right]=0(28)$
From the Routh-Hurwitz's criteria it is found that $E_{2}^{*}$ is locally asymptotically stable if the following conditions hold good.

$$
\begin{gather*}
A_{2} \beta_{20} \hat{y} /\left(B_{2}+\hat{y}\right)<D_{2}  \tag{29}\\
A_{1} \hat{y} /\left(B_{1}+\hat{x}\right)^{2}<a T_{0}\left(c+r_{0}\right) /\left(T_{0}+a \hat{x}\right)^{2} \tag{30}
\end{gather*}
$$

Condition (29) is satisfied automatically if

$$
\begin{equation*}
A_{2} \beta_{20}-D_{2}<0 \tag{31}
\end{equation*}
$$

Remark 2: From (31), it may be noted that if predation rate of top predator is less than the ratio of its death rate and its conversion efficiency then basal resource and intermediate consumer will survive and top predator goes to extinction.
4.The characteristic equation about $E_{3}^{*}$ is given by

$$
\begin{equation*}
\lambda^{3}+F_{1} \lambda^{2}+F_{2} \lambda+F_{3}=0 \tag{32}
\end{equation*}
$$

where,

$$
\begin{aligned}
& F_{1}=N_{11}-N_{22}, \\
& F_{2}=N_{11} N_{22}-N_{12} N_{21}-N_{23} N_{32}, \\
& F_{3}=N_{11} N_{23} N_{32}-N_{13} N_{21} N_{32}
\end{aligned}
$$

and

$$
\begin{aligned}
& N_{11}=\frac{A_{1} x^{*} y^{*}}{\left(B_{1}+^{*}\right)^{2}\left(1+z^{*}\right)}-\frac{a x^{*} T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a x^{*}\right)^{2}}, \\
& N_{12}=\frac{-A_{1} x^{*}}{\left(B_{1}+x^{*}\right)\left(1+z^{*}\right)}, N_{13}=\frac{A_{1} x^{*} y^{*}}{\left(B_{1}+x^{*}\right)\left(1+z^{*}\right)^{2}}, \\
& N_{21}=\frac{A_{1} B_{1} \beta_{10} y^{*}}{\left(B_{1}+x^{*}\right)^{2}\left(1+z^{*}\right)^{*}}, N_{22}=\frac{A_{2} y^{*} z^{*}}{\left(B_{2}+y^{*}\right)^{2}}, \\
& N_{23}=-\left(\frac{A_{1} \beta_{10} x^{*} y^{*}}{\left(B_{1}+x^{*}\right)\left(1+z^{*}\right)^{2}}+\frac{A_{2} y^{*}}{B_{2}+y^{*}}\right), \\
& N_{32}=\frac{A_{2} B_{2} \beta_{2 z^{z}}^{*}}{\left(B_{2}+y^{*}\right)^{2} .}
\end{aligned}
$$

According to Routh-Hurwitz criteria $E_{3}^{*}$ is locally asymptotically stable if $F_{1}>0, F_{2}>0, F_{3}>0$ and $F_{1} F_{2}>F_{3}$. It is difficult to interpret the results in ecological terms, from these complicated expressions, however, numerical examples are taken and graphs are plotted to illustrate the dynamical behaviour of the system about equilibrium $E_{3}^{*}$.

We are now in a position to make an attempt to find out the conditions under which the system undergoes Hopfbifurcation. For this purpose, we choose the parameter $A_{1}$ as bifurcation parameter as it plays a crucial role in Holling type II functional response which describes the predation of intermediate consumer. We shall now apply the Liu's criteria, (Liu, 1994) to obtain the conditions for small amplitude periodic solution arising from Hopfbifurcation.

As the equilibrium population densities are functions of $A_{1}$, the coefficients of the characteristic equation (32) are functions of the parameter $A_{1}$ and hence we can use the notation $F_{i}=F_{i}\left(A_{1}\right)$ for $i=1,2,3$. Noting that the quantities $F_{i} s$ are smooth functions of the parameter $A_{1}$, we first state in our case, the definition of a simple Hopf-Bifurcation.

If a critical value $A_{1}^{*}$ of parameter $A_{1}$ be found such that (i) a simple pair of complex conjugate eigenvalues of characteristic equation exists, say, $\lambda_{1}\left(A_{1}\right)=u\left(A_{1}\right)+i v\left(A_{1}\right)$, $\lambda_{2}\left(A_{1}\right)=u\left(A_{1}\right)-i v\left(A_{1}\right)=\overline{\lambda_{1}}\left(A_{1}\right)$. These eigen values will become purely imaginary at $A_{1}=A_{1}^{*}$, i.e., $\lambda_{1}\left(A_{1}^{*}\right)=i v_{0}, \lambda_{2}\left(A_{1}^{*}\right)=-i v_{0}$, with $v\left(A_{1}^{*}\right)=v_{0}>0$, and the other eigenvalue remains real and negative; and (ii) the transversality condition,
$d \operatorname{Re} \lambda_{i}\left(A_{1}^{*}\right) /\left.d A_{1}\right|_{A_{1}=A_{1}} ^{*}=d u\left(A_{1}\right) /\left.d A_{1}\right|_{A_{1}=A_{1}} ^{*} \neq 0$ is satisfied. Then we find at $A_{1}=A_{1}^{*}$, a simple Hopf-bifurcation. Without knowing eigenvalues, (Liu, 1994) proved that (referring the result to the current case): if $F_{1}\left(A_{1}\right), F_{3}\left(A_{1}\right), \Delta\left(A_{1}\right)=F_{1}\left(A_{1}\right) F_{2}\left(A_{1}\right)-F_{3}\left(A_{1}\right) \quad$ are smooth functions of the parameter ' $A_{1}$ ' in an open interval containing $A_{1}^{*} \varepsilon \Re^{+}$such that following conditions hold:

$$
\begin{aligned}
& \left(i_{*}\right) F_{1}\left(A_{1}^{*}\right)>0, \Delta\left(A_{1}^{*}\right)=0, F_{3}\left(A_{1}^{*}\right)>0 \\
& \left(i i_{*}\right) d \Delta\left(A_{1}\right) /\left.d A_{1}\right|_{A_{1}}=A_{1}^{*} \neq 0
\end{aligned}
$$

then $\left(i_{*}\right)$ and $\left(i i_{*}\right)$ are equivalent to conditions $(i)$ and (ii) for the occurrence of a simple Hopf-bifurcation at $A_{1}=A_{1}^{*}$. Hence we can propose the following theorem:
Theorem 3.1 If a critical value $A_{1}^{*}$ of parameter $A_{1}$ be found such that $F_{1}\left(A_{1}^{*}\right)>0, F_{3}\left(A_{1}^{*}\right)>0$ and $\Delta\left(A_{1}^{*}\right)=0$ and further $\Delta^{\prime} \neq 0$ (where prime denotes differentiation with respect to $A_{1}$ ) then system (11)-(13) undergoes Hopfbifurcation around $E_{3}^{*}$.
In the following theorem we show that the positive equilibrium is globally asymptotically stable. In order to prove this theorem we need the following lemma which establishes a region of attraction for Model 2.

Lemma 3.1 The set

$$
\Omega_{2}=\left\{(x, y, z): 0 \leq \beta_{10} x(t)+y(t)+z(t) / \beta_{20} \leq \kappa\right\}
$$

is a region of attraction for all solutions initiating in the interior of the positive region, where $\kappa=r_{0} T_{0} \beta_{10}\left(r_{0}+1\right) / a c \Phi \quad, \quad T_{0}=B\left(c+r_{0}-a\right) \quad$ and $\Phi=\min \left\{1, D_{1}, D_{2}\right\}$
Proof: From Eq. (11) we get,

$$
d x / d t \leq x\left[r_{0} T_{0}-a c x\right] / T_{0}
$$

then by the usual comparison theorem, we get as $t \rightarrow \infty$,

$$
\begin{equation*}
x \leq r_{0} T_{0} / a c \tag{33}
\end{equation*}
$$

where $T_{0}=B\left(c+r_{0}-a\right)$.
Now, let us consider the following function:
$W(t)=\beta_{10} x(t)+y(t)+z(t) / \beta_{20}$
by using Eqs. (12), (13) and (33) we get

$$
d w / d t+\Phi W \leq \beta_{10} r_{0} T_{0}\left(r_{0}+1\right) / a c
$$

Where $\Phi=\min \left\{1, D_{1}, D_{2}\right\}$
and then by the usual comparison theorem, we get as
$t \rightarrow \infty, W(t) \leq \beta_{10} r_{0} T_{0}\left(r_{0}+1\right) / \Phi a c$ and hence,

$$
\beta_{10} x(t)+y(t)+z(t) / \beta_{20^{\prime}} \leq \kappa
$$

where $\kappa=\beta_{10} r_{0} T_{0}\left(r_{0}+1\right) / \Phi a c$
This proves the lemma.
Theorem 3.2 Let the following inequalities hold in the region $\Omega_{2}$.
$\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}<\frac{a\left(c+r_{0}\right)}{T_{0}+a x^{*}}$,
$\frac{A_{1} \beta_{10} x^{*}}{B_{1}+x^{*}}<\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\kappa\right)\left(B_{2}+y^{*}\right)}+D_{1}$,
$\left[\frac{A_{1} \beta_{10}{ }^{\kappa}}{B_{1}+x^{*}}-\frac{A_{1}}{(1+\kappa)\left(B_{1}+\kappa\right)}\right]^{2}$
$<\left[\left(\frac{a T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a \kappa\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}\right)\right.$
$\left.\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\kappa\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{10} x^{*}}{B_{1}+x^{*}}\right)\right]$,

$<G_{1}\left[\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\kappa\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{10} x^{*}}{B_{1}+x^{*}}\right)\left(D_{2}-\frac{A_{2} \beta_{20} y^{*}}{B_{2}+y^{*}}\right)\right]$,
(37)
where,
$G_{1}>\left[\frac{\left(\frac{A_{1} y^{*}}{\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right)^{2}}{\left(\frac{a T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a \kappa\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}\right)\left(D_{2}-\frac{A_{2} \beta_{20} y^{*}}{B_{2}+y^{*}}\right)}\right]$
(38)
then the positive equilibrium $E_{3}^{*}$ is globally asymptotically stable with respect to all solutions initiating in the interior of positive region $\Omega_{2}$. (for proof see Appendix).

## IV. ANALYSIS OF MODEL 1

### 4.1 EQUILIBRIA OF MODEL 1

The Model 1 has following four non-negative equilibria. They are listed below:

$$
\begin{aligned}
& \text { 1. } E_{0 w}^{*}=(0,0,0,0,0) \\
& \text { 2. } E_{1 w}^{*}=\left(\tilde{x}, 0,0, \tilde{C_{0}}, \tilde{C_{E}}\right) \\
& \tilde{x}=r\left(\tilde{C_{0}}\right) T_{0} / a c>0 \\
& \text { if } r\left(\tilde{C_{0}}\right)>0,
\end{aligned}
$$

$\tilde{C_{E}}=\left(g_{0}\left(l_{1}+l_{2}\right)+d_{1} \theta \beta k_{1} l_{1} \tilde{x} / a_{1}\right) /\left(\left(k_{1} a_{1} \tilde{x}+k_{2}\right) l_{2}+k_{2} l_{1}\right)$ and the $\tilde{C_{0}}$ is given by the positive root of the equation

$$
\begin{equation*}
K_{1}{\tilde{C_{0}}}^{2}-K_{2} \tilde{C_{0}}+K_{3}=0 \tag{41}
\end{equation*}
$$

where,
$K_{1}=r_{1} k_{1} l_{2} a_{1}^{2} T_{0} / a c$,
$K_{2}=\left(k_{1} l_{2} a_{1}^{2} r_{0}^{2} T_{0}+a_{1} k_{2}\left(l_{1}+l_{2}\right) a c+a_{1} r_{1} k_{1} d_{1} \theta \beta T_{0}\right) / a c$,
$K_{3}=g_{0} a_{1}^{2}+\left(a_{1} r_{0} k_{1} T_{0} / a c+k_{2}\right) d_{1} \theta \beta$.
3. $E_{2 w}^{*}=\left(\hat{x}, \hat{y}, 0, \hat{C_{0}}, \hat{C_{E}}\right)$

Now, we show the existence of $E_{2 w}^{*}$ as follows:
Here $\hat{x}, \hat{y}, \hat{C_{0}}$ and $\hat{C_{E}}$ are the positive solutions of the system of algebraic equations given below:
(3.1a) $C_{0}=\frac{g_{0} a_{1}^{2}+\left(k_{1} a_{1} x+k_{2}\right) d_{1} \theta \beta}{k_{1} l_{2} a_{1}^{2} x+k_{2} a_{1}\left(l_{1}+l_{2}\right)}=g_{1}(x)$,

(3.1c)
$y=\frac{B_{1} \beta_{1}\left(C_{0}\right)\left[r\left(C_{0}\right) T_{0}\left(A_{1} \beta_{1}\left(C_{0}\right)-D_{1}\right)-a c B_{1} D_{1}\right]}{\left(A_{1} \beta_{1}\left(C_{0}\right)-D_{1}\right)\left[T_{0}\left(A_{1} \beta_{1}\left(C_{0}\right)-D_{1}\right)+a B_{1} D_{1}\right]}=g_{3}(x)$,
substituting the values of $y$ and $C_{0}$ in $B_{1}+x=\frac{A_{1} y\left(T_{0}+a x\right)}{r\left(C_{0}\right) T_{0}-a c x}$, we get
(3.1d) $x=\frac{A_{1} g_{3}(x)\left(T_{0}+a g_{3}(x)\right)}{\left(r_{0}-r_{1} g_{1}(x)\right) T_{0}-a c x}-B_{1}$

Let (3.2a) $M(x)=\frac{A_{1} g_{3}(x)\left(T_{0}+a g_{3}(x)\right)}{\left(r_{0}-r_{1} g_{1}(x)\right) T_{0}-a c x}-\left(B_{1}+x\right)$
To show the existence of $E_{2 w}^{*}$, it suffices to show that equation (3.2a) has a unique positive solution for this we may note that
(3.2b) $\left.M(0)=\frac{A_{1} g_{3}(0)}{r_{0}-r_{1} g_{1}(0)}\right)-B_{1}>0$
(3.2c) $M\left(s_{0}\right)=-\left[\frac{A_{1} g_{3}\left(s_{0}\right)\left(T_{0}+a s_{0}\right)}{r_{1} g_{1}\left(s_{0}\right) T_{0}}+B_{1}+s_{0}\right]<0$
where, $x=s_{0}=r_{0} T_{0} / a c$ and $\lim _{x \rightarrow s_{0}} M(x) \rightarrow \infty$.
for the unique positive solution $x$, we must have
(3.2d) $d M / d x=\left(M_{2}\left(d M_{1} / d x\right)+M_{1}\left(d M_{2} / d x\right)-M_{2}^{2}\right) / M_{2}^{2}<0$
where, $M_{1}=A_{1} g_{3}(x)\left(T_{0}+a x\right)$ and
$M_{2}=\left(r_{0}-r_{1} g_{1}(x)\right) T_{0}-a c x$.
Knowing the value of $\hat{x}$, the values of $\hat{C_{0}}, \hat{C_{E}}$ and $\hat{y}$ can be computed from equations (3.1a), (3.1b) and (3.1c) respectively.
4. $E_{3 w}^{*}=\left(x^{*}, y^{*}, z^{*}, C_{0}^{*}, C_{E}^{*}\right)$

Now, we show the existence of $E_{3 w}^{*}$ as follows:
Here $x^{*}, y^{*}, z^{*}, C_{0}^{*}$ and $C_{E}^{*}$ are the positive solutions of the system of algebraic equations given below:
(4.1a) $C_{0}=\frac{g_{0} a_{1}^{2}+\left(k_{1} a_{1} x+k_{2}\right) d_{1} \theta \beta}{k_{1} l_{2} a_{1}^{2} x+k_{2} a_{1}\left(l_{1}+l_{2}\right)}=f_{1}(x)$,
$g_{0}\left(l_{1}+l_{2}\right)+\frac{d_{1}}{a_{1}} \theta \beta k_{1} l_{1} x$
(4.1b) $C_{E}=\frac{1}{\left(k_{1} a_{1} x+k_{2}\right) l_{2}+k_{2} l_{1}}=f_{2}(x)$,
(4.1c) $y=\frac{B_{2} D_{2}}{A_{2} \beta_{2}\left(C_{0}\right)-D_{2}}=f_{3}(x)$,
(4.1d)
$z=\frac{\beta_{1}\left(C_{0}\right) \beta_{2}\left(C_{0}\right) x\left(r\left(C_{0}\right) T_{0}-a c x\right)}{D_{2}\left(\left(T_{0}+a x\right)\right)}-\frac{\beta_{2}\left(C_{0}\right) D_{1} y}{D_{2}}=f_{4}(x)$
substituting the values of $y, z$ and $C_{0}$ in $B_{1}+x=\frac{A_{1} y\left(T_{0}+a x\right)}{(1+z)\left[r\left(C_{0}\right) T_{0}-a c x\right]}$, we get
(4.1e) $x=\frac{A_{1} f_{3}(x)\left(T_{0}+a x\right)}{\left(1+f_{4}(x)\right)\left[\left(r_{0}-r_{1} f_{1}(x)\right) T_{0}-a c x\right]}-B_{1}$

Let
(4.2a) $F(x)=\frac{A_{1} f_{3}(x)\left(T_{0}+a x\right)}{\left(1+f_{4}(x)\right)\left[\left(r_{0}-r_{1} f_{1}(x)\right) T_{0}-a c x\right]}-\left(B_{1}+x\right)$

To show the existence of $E_{3 w}^{*}$, it suffices to show that equation (4.2a) has a unique positive solution for this we may note that
(4.2b) $F(0)=\frac{A_{1} f_{3}(0)}{\left[1+f_{4}(0)\right]\left(r_{0}-r_{1} f_{1}(0)\right)}-B_{1}>0$
(4.2c) $F\left(k_{0}\right)=-\left[\frac{A_{1} f_{3}\left(k_{0}\right)\left(T_{0}+a k_{0}\right)}{\left(1+f_{4}\left(k_{0}\right)\right) r_{1} f_{1}\left(k_{0}\right) T_{0}}+B_{1}+k_{0}\right]<0$
$\lim$
where, $x=k_{0}=r_{0} T_{0} / a c$ and $\lim _{x \rightarrow k_{0}} F(x) \rightarrow \infty$.
for the unique positive solution $x$, we must have (4.2d)
$M 2=(r 0 r 1 g 1(x)) T 0 a c x$.
where, $G=A_{1} f_{3}(x)\left(T_{0}+a x\right) /\left(1+f_{4}(x)\right)$
Knowing the value of $x^{*}$, the values of $C_{0}^{*}, C_{E}^{*}, y^{*}$ and $z^{*}$ can be computed from equations (4.1a), (4.1b), (4.1c) and (4.1d) respectively.
where, $G=A_{1} f_{3}(x)\left(T_{0}+a x\right) /\left(1+f_{4}(x)\right)$
Knowing the value of $x^{*}$, the values of $C_{0}^{*}, C_{E}^{*}, y^{*}$ and $z^{*}$ can be computed from equations (4.1a), (4.1b), (4.1c) and (4.1d) respectively.

### 4.2 Dynamical behaviour of Model 1

The general variational matrix corresonding to the Model 1:

$$
\left.J\left(x, y, z, C_{0}, C_{E}\right)=\left[\begin{array}{cccc}
v_{11} & v_{12} & v_{13} & v_{14}
\end{array}\right]\left[\begin{array}{cccc}
v_{21} & v_{22} & v_{23} & v_{24}
\end{array}\right] \begin{array}{cccc}
0 & v_{32} & v_{33} & v_{34} \\
0 & 0 & 0 & v_{44} \\
v_{45} \\
v_{51} & 0 & 0 & v_{54} \\
v_{55}
\end{array}\right]
$$

where,
$v_{11}=\frac{r\left(C_{0}\right) T_{0}^{2}-2 a c x T_{0}-a^{2} c x^{2}}{\left(T_{0}+a x\right)^{2}}-\frac{A_{1} B_{1} y}{\left(B_{1}+x\right)^{2}(1+z)}$,
$v_{12}=\frac{-A_{1} x}{\left(B_{1}+x\right)(1+z)}, v_{13}=\frac{A_{1} x y}{\left(B_{1}+x\right)(1+z)^{2}}$,
$v_{14}=\frac{x T_{0} r^{\prime}\left(C_{0}\right)}{T_{0}+a x}, v_{21}=\frac{A_{1} B_{1} \beta_{1}\left(C_{0}\right) y}{\left(B_{1}+x\right)^{2}(1+z)}$,
$v_{22}=\frac{A_{1} \beta_{1}\left(C_{0}\right) x}{\left(B_{1}+x\right)(1+z)}-\frac{A_{2} B_{2} z}{\left(B_{2}+y\right)^{2}}-D_{1}$,
$v_{23}=-y\left[\frac{A_{1} \beta_{1}\left(C_{0}\right) x}{\left(B_{1}+x\right)(1+z)^{2}}+\frac{A_{2}}{B_{2}+y}\right], v_{24}=\frac{A_{1} \beta_{1}{ }^{\prime}\left(C_{0}\right) x y}{\left(B_{1}+x\right)(1+z)}$,
$v_{32}=\frac{A_{2} B_{2} \beta_{2}\left(C_{0}\right) z}{\left(B_{2}+y\right)^{2}}, v_{33}=\frac{A_{2} \beta_{2}\left(C_{0}\right) y}{B_{2}+y}-D_{2}$,
$v_{34}=\frac{A_{2} \beta_{2}^{\prime}\left(C_{0}\right) y z}{B_{2}+y}, v_{44}=-\left(l_{1}+l_{2}\right), v_{45}=a_{1}$,
$v_{51}=k_{1}\left(l_{1} C_{0}-a_{1} C_{E}\right), v_{54}=k_{1} l_{1} x, v_{55}=-\left(k_{1} a_{1} x+k_{2}\right)$.

1. At $E_{0 w}^{*}$, the eigenvalues of the characteristic equation are $r_{0},-D_{1},-D_{2},-\left(l_{1}+l_{2}\right)$ and $-k_{2}$, showing the instability of $E_{0 w}^{*}$.
2. At $E_{1 w}^{*}$, two eigenvalues of the characteristic equation are, $A_{1} \beta_{1}\left(\tilde{C_{0}}\right) \tilde{x} /\left(B_{1}+\tilde{x}\right)-D_{1},-D_{2}$ and the other three eigenvalues are given by the roots of the following cubic equation

$$
\begin{equation*}
\lambda^{3}+\lambda^{2} S_{1}+\lambda S_{2}+S_{3}=0 \tag{42}
\end{equation*}
$$

where,
$S_{1}=\left[l_{1}+l_{2}+k_{2}+\frac{k_{1} a_{1} T_{0} r\left(\tilde{C_{0}}\right)}{a c}+\frac{c r\left(\tilde{C_{0}}\right)}{c+r\left(\tilde{C_{0}}\right)}\right]$,
$S_{2}=\frac{c r\left(\tilde{C_{0}}\right)}{c+r\left(\tilde{C_{0}}\right)}\left[l_{1}+l_{2}+k_{2}+k_{1} a_{1} T_{0} r\left(\tilde{C_{0}}\right) / a c\right]$
$+\left[l_{1} k_{2}+l_{2}\left(k_{2}+k_{1} a_{1} T_{0} r\left(\tilde{C_{0}}\right) / a c\right)\right]$,
$S_{3}=\frac{r\left(\tilde{C_{0}}\right)}{a\left(c+r\left(\tilde{C_{0}}\right)\right)}\left[a c l_{1} k_{2}+k_{1}\left(a_{1} l_{2} T_{0} r\left(\tilde{C_{0}}\right)\right.\right.$
$\left.\left.+r_{1} d_{1} \theta \beta\right)+l_{2}\left(a c k_{2}-a_{1} r_{1} k_{1} \tilde{C_{0}}\right)\right]$.

According to Routh-Hurwitz criteria $E_{1 w}^{*}$ is locally asymptotically stable if

$$
\begin{align*}
& A_{1} \beta_{1}\left(\tilde{C_{0}}\right) \tilde{x} /\left(B_{1}+\tilde{x}\right)<D_{1},  \tag{43}\\
& \tilde{C_{0}}<a c k_{2} / a_{1} r_{1} k_{1}
\end{align*}
$$

and $S_{1} S_{2}-S_{3}>0$ which implies
$J_{1} J_{3}\left(J_{1}+J_{3}\right)+J_{3}\left(l_{1} k_{2}+l_{2} J_{2}\right)+k_{1} r_{1}\left(a_{1} l_{2} \tilde{C_{0}}-d_{1} \theta \beta\right) / a>0$
where $J_{1}=c r\left(\tilde{C_{0}}\right) /\left(c+r\left(\tilde{C_{0}}\right)\right), \quad J_{2}=k_{2}+k_{1} a_{1} T_{0} r\left(\tilde{C_{0}}\right) / a c$ and $J_{3}=l_{1}+l_{2}+J_{2}$.
Condition given in equation (44) is satisfied if $\tilde{C_{0}}>d_{1} \theta \beta / a_{1} l_{2}$

We note from (43) that, $E_{1 w}^{*}$ of the system is locally asymptotically stable if the condition

$$
\begin{equation*}
r\left(\tilde{C_{0}}\right) T_{0}\left(A_{1} \beta_{1}\left(\tilde{C_{0}}\right)-D_{1}\right)<a c B_{1} D_{1} \tag{45}
\end{equation*}
$$

is being satisfied. The condition (45) is automatically satisfied if

$$
\begin{equation*}
A_{1} \beta_{1}\left(\tilde{C_{0}}\right) \leq D_{1} \tag{46}
\end{equation*}
$$

In this case also similar result can be stated as mentioned in Remark 1.
3.At $E_{2 w}^{*}$, one of the eigenvalues of the characteristic equation is $A_{2} \beta_{2}\left(\hat{C_{0}}\right) \hat{y} /\left(B_{2}+\hat{y}\right)-D_{2}$ and the other four eigenvalues are given by the roots of the following equation is

$$
\begin{equation*}
\lambda^{4}+\lambda^{3} T_{1}+\lambda^{2} T_{2}+\lambda T_{3}+T_{4}=0 \tag{47}
\end{equation*}
$$

where,
$T_{1}=\left(l_{1}+l_{2}+k_{1} a_{1} \hat{x}+k_{2}\right)-\frac{A_{1} \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{2}}+\frac{a T_{0}\left(c+r\left(\hat{C_{0}}\right)\right) \hat{x}}{\left(T_{0}+a \hat{x}\right)^{2}}$,
$T_{2}=l_{1} k_{2}+l_{2}\left(k_{1} a_{1} \hat{x}+k_{2}\right)$
$+\left(k_{1} a_{1} \hat{x}+k_{2}+l_{1}+l_{2}\right)\left(\frac{A_{1} \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{2}}-\frac{a T_{0}\left(c+r\left(\hat{C_{0}}\right)\right) \hat{x}}{\left(T_{0}+a \hat{x}\right)^{2}}\right)$
$+\frac{A_{1}^{2} B_{1} \beta_{1}\left(\hat{C_{0}}\right) \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{3}} T_{3}=\left(k_{1} a_{1} \hat{x}+k_{2}+l_{1}+l_{2}\right) \frac{A_{1}^{2} B_{1} \beta_{1}\left(\hat{C_{0}}\right) \hat{x} \hat{y}}{\left(B_{1}+\hat{x}\right)^{3}}$
$+\left(l_{1}+l_{2}\right)\left(l_{1} \hat{C_{0}}-a_{1} \hat{C_{E}}\right) \frac{k_{1} A_{1}^{2} \beta_{1}^{\prime}\left(\hat{C_{0}}\right)(\hat{x})^{2} \hat{y}}{\left(B_{1}+\hat{x}\right)^{2}}$.
From the Routh-Hurwitz's criteria it is found that $E_{2 w}^{*}$ is locally asymptotically stable if the following conditions hold good.

$$
\begin{equation*}
A_{2} \beta_{2}\left(\hat{C_{0}}\right) \hat{y} /\left(B_{2}+\hat{y}\right)<D_{2}, \tag{48}
\end{equation*}
$$

$T_{i}>0, i=1,2,3,4, T_{1} T_{2}>T_{3}$ and $T_{1} T_{2} T_{3}>\left(T_{3}^{2}+T_{1}^{2} T_{4}\right)$.
Condition (48) is satisfied automatically if

$$
\begin{equation*}
A_{2} \beta_{2}\left(\hat{C_{0}}\right)<D_{1} \tag{49}
\end{equation*}
$$

In this case similar result is observed as given in Remark 2.
4.The characteristic equation of $E_{3 w}^{*}$ is

$$
\begin{equation*}
\lambda^{5}+\lambda^{4} W_{1}+\lambda^{3} W_{2}+\lambda^{2} W_{3}+\lambda W_{4}+W_{5}=0 \tag{50}
\end{equation*}
$$

where,
$W_{1}=l_{1}+l_{2}+k_{1} a_{1} x^{*}+k_{2}-P_{1} y^{*} z^{*}-x^{*}\left(P_{2} P_{3} y^{*}-a P_{4}\left(c+r\left(C_{0}^{*}\right)\right)\right)$,
$W_{2}=\left(l_{1} k_{2} l_{2}\left(k_{1} a_{1} x^{*}+k_{2}\right)\right)-\left(k_{1} a_{1} x^{*}+k_{2}\right)\left(P_{1} y^{*} z^{*}-x^{*}\left(P_{2} P_{3} y^{*}-a P_{4}\left(c+r\left(C_{0}^{*}\right)\right)\right)\right)$

$\left.+P_{1}\left(B_{2}+y^{*}\right)\right) P_{1} B_{2} \beta_{2}\left(C_{0}^{*}\right) z^{*}+P_{2}^{2} P_{3} B_{1} \beta_{1}\left(C_{0}\right) x^{*} y^{*}$
$W_{3}=\left(l_{1}+l_{2}+k_{1} a_{1} x^{*}+k_{2}\right)\left[P_{1} x^{*} y^{*} z^{*}{ }^{*}\left(P_{2} P_{3} y^{*}{ }^{*}-a P_{4}\left(c+r\left(C_{0}{ }^{*}\right)\right)\right)\right.$
$\left.+\left(P_{2} P_{5} \beta_{1}\left(C_{0}^{*}\right) x^{*}+P_{1}\left(B_{2}+y^{*}\right)\right) P_{1} B_{2} \beta_{2}\left(C_{0}^{*}\right) y^{*} z^{*}+P_{2}^{2} P_{3} B_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*} y^{*}\right]$
$-\left(P_{2} P_{3} y^{*}-a P_{4}\left(c+r\left(C_{0}^{*}\right)\right)\right)\left(P_{2} P_{5} \beta_{1}\left(C_{0}^{*}\right) x^{*}+P_{1}\left(B_{2}+y^{*}\right)\right) P_{1} B_{2} \beta_{2}\left(C_{0}^{*}\right) x^{*} y^{*} z^{*}$
$-P_{1} P_{2}^{2} B_{2} P_{3} P_{5} \beta_{1}\left(C_{0}^{*}\right) \beta_{2}\left(C_{0}^{*}\right) x^{*} y^{*} z^{*}+a_{1} k_{1} r_{1} x^{*} P_{4}\left(T_{0}+a x^{*}\right)\left(l_{1} l_{0}^{*}-a_{1} C_{E}^{*}\right)$
$-\left[x^{*}\left(P_{2} P_{3} y^{*}-a P_{4}\left(c+r\left(C_{0}^{*}\right)\right)\right)+P_{1} y^{*} z^{*}\right]\left(l_{1} k_{2}+l_{2}\left(k_{1} a_{1} x^{*}+k_{2}\right)\right)$
$W_{4}=\left(l l_{2}+l_{2}\left(k a x^{*}+k_{2}\right)\right) y^{*}\left(P_{2} P_{\beta} \beta_{1}\left(C_{0}^{*}\right) x^{*}+P_{1}\left(B_{2}+y^{*}\right)\right) P P_{1} \beta_{2}\left(C_{0}\right) z^{*}$
$\left.+\left(P_{2} P_{3} y^{*}-a P_{4}\left(c+r\left(C_{0}^{*}\right)\right)\right) P_{1} x^{2} y^{*} z^{*}+P_{2}^{2} P_{3} B_{1} \beta_{1}{ }_{1}\left(C_{0}^{*}\right) x^{*} y^{*}\right]$
and $\quad P_{1}=A_{2} /\left(B_{2}+y^{*}\right)^{2}, P_{2}=A_{1} /\left(\left(B_{1}+x^{*}\right)\left(1+z^{*}\right)\right)$
$P_{3}=1 /\left(B_{1}+x^{*}\right), P_{4}=T_{0} /\left(T_{0}+a x^{*}\right)^{2}, P_{5}=1 /\left(1+z^{*}\right)$.

According to Routh-Hurwitz's criteria, the equilibrium point $E_{3 w}^{*}$ is locally asymptotically stable if $\quad W_{i}>0, \quad i=1,2,3,4,5, \quad W_{1} W_{2}>W_{3}$, $W_{1} W_{2} W_{3}>\left(W_{3}^{2}+W_{1}^{2} W_{4}\right)$ $\left(W_{3} W_{4}-W_{2} W_{5}\right)\left(W_{1} W_{2}-W_{3}\right)>\left(W_{1} W_{4}-W_{5}\right)^{2}$. and It is difficult to interpret the results in ecological terms from these complicated expressions, however, numerical examples are taken and graphs are ploted to illustrate the dynamical behaviour of the system about equilibrium $E_{3 w}^{*}$.

Again, in the similar way the equilibrium population densities are functions of $A_{1}$ and the coefficients of the characteristic equation (50) are functions of the parameter $A_{1}$. Now we can use the notation $W_{i}=W_{i}\left(A_{1}\right)$ for $i=1,2,3,4,5$. Now noting that the quantities $W_{i}$ are smooth functions of the parameter $A_{1}$. As we have explained the definition of Hopf-bifurcation in previous section. Without knowing eigenvalues, (Liu, 1994) proved that (referring the result to the current case):if $W_{i}\left(A_{1}\right)$.
$\Delta_{1}\left(A_{1}\right)=W_{1}\left(A_{1}\right) W_{2}\left(A_{1}\right)-W_{3}\left(A_{1}\right)$.
$\Delta_{2}\left(A_{1}\right)=W_{1}\left(A_{1}\right) W_{2}\left(A_{1}\right) W_{3}\left(A_{1}\right)-\left(W_{3}^{2}\left(A_{1}\right)+W_{1}^{2}\left(A_{1}\right) W_{4}\left(A_{1}\right)\right)$.
$\Delta_{3}\left(A_{1}\right)=\left[W_{3}\left(A_{1}\right) W_{4}\left(A_{1}\right)-W_{2}\left(A_{1}\right) W_{5}\left(A_{1}\right)\right]\left[W_{1}\left(A_{1}\right) W_{2}\left(A_{1}\right)-W_{3}\left(A_{1}\right)\right]$
$-\left[W_{1}\left(A_{1}\right) W_{4}\left(A_{1}\right)-W_{5}\left(A_{1}\right)\right]^{2}$
are smooth functions of the parameter ' $A_{1}$ ' in an open interval containing $A_{1}^{*} \varepsilon \mathfrak{R}^{+}$such that following conditions hold:
$($ iiii $) W_{1}\left(A_{1}^{*}\right)>0, \Delta_{1}\left(A_{1}^{*}\right)>0, \Delta_{2}\left(A_{1}^{*}\right)>0$ and $\Delta_{3}\left(A_{1}^{*}\right)=0$;
$\left(i v_{*}\right) d \Delta_{3}\left(A_{1}\right) /\left.d A_{1}\right|_{A_{1}=A_{1}} * \neq 0$
then $\left(i i i_{*}\right)$ and ( $i v_{*}$ ) are equivalent to conditions (i) and (ii) mentioned in section 3.2, for the occurrence of a simple Hopf-bifurcation at $A_{1}=A_{1}^{*}$. Hence, in the similar way, we can propose the following theorem:

Theorem 4.1 If a critical value $A_{1}^{*}$ of parameter $A_{1}$ be found such that $W_{i}\left(A_{1}^{*}\right)>0, \Delta_{1}\left(A_{1}^{*}\right)>0, \Delta_{2}\left(A_{1}^{*}\right)>0, \Delta_{3}\left(A_{1}^{*}\right)=0$ and further $\Delta_{3}^{\prime} \neq 0$ (where primes denotes differentiation with respect to $A_{1}$ ) then system (1)(5) undergoes Hopf-bifurcation around $E_{3 w}^{*}$.

In the following theorem we show that the positive equilibrium is globally asymptotically stable. In order to prove this theorem we need the following lemma which establishes a region of attraction for Model 1.

Lemma 4.1 All the solutions of Model 1, will lie in the region $\Omega_{1}$, where
$\Omega_{1}=\left\{\left(\mathrm{x}, \mathrm{y}, \mathrm{z}, \mathrm{C}_{0}, \mathrm{C}_{\mathrm{E}}\right): \quad 0 \leq \beta_{1}\left(\mathrm{C}_{01}\right) \mathrm{x}(\mathrm{t})\right)+\mathrm{y}(\mathrm{t})+\mathrm{z}(\mathrm{t}) /$ $\left.\beta_{2}\left(\mathrm{C}_{01}\right) \leq \Psi_{1}, 0 \leq \mathrm{C}_{0}(\mathrm{t})+\mathrm{C}_{\mathrm{E}}(\mathrm{t}) \leq \Psi_{2}\right\}$,
where $\psi_{1}=T_{0} r\left(C_{0 l}\right) \beta_{1}\left(C_{0 l}\right)\left(r\left(C_{0 l}\right)+1\right) / a c \Phi_{1}, \psi_{2}=\mu_{1} / \nu_{1}$,
$T_{0}=B\left(c+r_{0}-a\right), \Phi_{1}=\min \left\{1, D_{1}, D_{2}\right\}$
$\mu_{1}=g_{0}+d_{1} \theta \beta / a_{1}$ and
$v_{1}=\left\{k_{2}-a_{1},\left(l_{1}+l_{2}\right)-k_{1} l_{1} T_{0} r\left(C_{0 l}\right) / a c\right\}$.

Proof: From Eq. (4) we get,

$$
d C_{0} / d t \geq d_{1} \theta \beta / a_{1}-\left(l_{1}+l_{2}\right) C_{0}
$$

then by the usual comparison theorem, we get as $t \rightarrow \infty$

$$
C_{01} \geq d_{1} \theta \beta /\left(a_{1}\left(l_{1}+l_{2}\right)\right)
$$

From Eq. (1) we get,

$$
d x / d t \leq x\left(r\left(C_{0}\right) T_{0}-a c x\right) / T_{0}
$$

then by the usual comparison theorem, we get as $t \rightarrow \infty$

$$
\begin{equation*}
x \leq r\left(C_{0 l}\right) T_{0} / a c \tag{51}
\end{equation*}
$$

where $T_{0}=B\left(c+r_{0}-a\right)$

Now, let us consider the following function:

$$
w(t)=\beta_{1}\left(C_{0 l}\right) x(t)+y(t)+z(t) / \beta_{2}\left(C_{0 l}\right)
$$

by using Eqs. (2), (3) and (51) we get
$d w / d t+\Phi_{1} w \leq T_{0} r\left(C_{0 l}\right) \beta_{1}\left(C_{0 l}\right)\left(r\left(C_{0 l}\right)+1\right) / a c$
where $\Phi_{1}=\min \left\{1, D_{1}, D_{2}\right\}$
then by the usual comparison theorem, we get as $t \rightarrow \infty, w \leq T_{0} r\left(C_{0 l}\right) \beta_{1}\left(C_{0 l}\right)\left(r\left(C_{0 l}\right)+1\right) / a c \Phi_{1}$ and hence

$$
\beta_{1}\left(C_{0 l}\right) x(t)+y(t)+z(t) / \beta_{2}\left(C_{0 l}\right) \leq \psi_{1}
$$

where $\psi_{1}=T_{0} r\left(C_{0 l}\right) \beta_{1}\left(C_{0 l}\right)\left(r\left(C_{0 l}\right)+1\right) / a c \Phi_{1}$
Finally, let us consider the following function:

$$
w_{1}(t)=C_{0}(t)+C_{E}(t)
$$

by using Eqs. (4), (5) and (51) we get,
$\frac{d w_{1}}{d t} \leq\left(g_{0}+d_{1} \theta \beta / a_{1}\right)-\left[\left(l_{1}+l_{2}\right)-k_{1} l_{1} T_{0} r\left(C_{0 l}\right) / a c\right] C_{0}-\left(k_{2}-a_{1}\right) C_{E}$
if $k_{2}>a_{1},\left(l_{1}+l_{2}\right)>k_{1} l_{1} T_{0} r\left(C_{0 l}\right) / a c$

$$
d w_{1} / d t+v_{1} w_{1} \leq \mu_{1}
$$

where $\quad \mu_{1}=g_{0}+d_{1} \theta \beta / a_{1}$
$v_{1}=\left\{k_{2}-a_{1},\left(l_{1}+l_{2}\right)-k_{1} l_{1} T_{0} r\left(C_{0 l}\right) / a c\right\}$
then, by the usual comparison theorem, we get as $t \rightarrow \infty$
$w_{1}(t) \leq \mu_{1} / \nu_{1}=\psi_{2}$ and hence

$$
C_{0}(t)+C_{E}(t) \leq \psi_{2}
$$

This proves the lemma.

Theorem 4.2 Let the following inequalities hold in the region $\Omega_{1}$ :
$\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}<\frac{a T_{0}\left(c+r\left(C_{0}^{*}\right)\right)}{\left(T_{0}+a \psi_{1}\right)\left(T_{0}+a x^{*}\right)}$,
$\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*}}{B_{1}+x^{*}}<\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\psi_{1}\right)\left(B_{2}+y^{*}\right)}+D_{1}$,

$$
\begin{aligned}
{\left[\frac{A_{1} E_{1} \beta_{1}\left(C_{0}^{*}\right) \psi_{1}}{B_{1}+x^{*}}\right.} & \left.-\frac{A_{1}}{\left(1+\psi_{1}\right)\left(B_{1}+\psi_{1}\right)}\right]^{2} \\
& <\frac{E_{1}}{3}\left[\left(\frac{a T_{0}\left(c+r\left(C_{0}^{*}\right)\right)}{\left(T_{0}+a \psi_{1}\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}\right)\right.
\end{aligned}
$$

$$
\begin{equation*}
\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\psi_{1}\right)\left(B_{2}+y^{*}\right)^{*}}+D_{1}-\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*}}{B_{1}+x^{*}}\right) \mathrm{l} \tag{54}
\end{equation*}
$$

$\left[\frac{A_{2} E_{2} \beta_{2}\left(C_{0}^{*}\right) \psi_{1}}{B_{2}+y^{*}}-\frac{A_{1} E_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*} y^{*}}{\left(1+\psi_{1}\right)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right]^{2}$
$<\frac{4 E E_{2}}{9}\left[\left(\frac{A_{2} B z^{*}}{\left(B_{2}+\psi_{1}\right)\left(B_{2}+y^{*}\right)^{*}}+D_{1}-\frac{A \beta_{1}\left(C_{0}^{*}\right) x}{B_{1}+x^{*}}\right)\left(D_{2}-\frac{A_{2} \beta_{2}\left(C_{0}^{*}\right) y^{*}}{B_{2}+y^{*}}\right)\right]$,
$\left[a_{1} E_{3}+E_{4} k_{1} l_{1} \psi_{1}\right]^{2}<\frac{E_{3} E_{4}}{2}\left[\left(k_{1} a_{1} x^{*}+k_{2}\right)\left(l_{1}+l_{2}\right)\right]$
where
$E_{E}<\left(\frac{E_{3}\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+\psi_{1}\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*}}{B_{1}+x^{*}}\right)\left(l_{1}+l_{2}\right)}{4\left(\frac{A_{1} \alpha\left(\psi_{2}\right) \psi_{1}^{2}}{B_{1}}\right)^{2}}\right)$
and $\quad E_{2}>\left(\frac{3\left(\frac{A_{1} y^{*}}{\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right)^{2}}{\left(\frac{a T_{0}\left(c+r\left(C_{0}^{* *}\right)\right)}{\left(T_{0}+a \psi_{1}\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}\right)\left(D_{2}-\frac{A_{2} \beta_{2}\left(C_{0}^{*}\right) y^{* *}}{B_{2}+y^{*}}\right)}\right)$

and

$$
\begin{equation*}
E_{4}<\left(\frac{\left(\frac{a T_{0}\left(c+r\left(C_{0}^{*}\right)\right)}{\left(T_{0}+a \psi_{1}\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{B_{1}\left(B_{1}+x^{*}\right)}\right)\left(k_{1} a_{1} x^{*}+k_{2}\right)}{2\left(k_{1} l_{1} C_{0}^{*}\right)^{2}}\right) \tag{60}
\end{equation*}
$$

then the positive equilibrium $E_{3 w}^{*}$ is globally asymptotically stable with respect to all solutions initiating in the interior of the positive region $\Omega_{1}$. (for proof see Appendix).

## V. NUMERICAL EXAMPLE

In this section, we demonstrate the dynamical behavior of a three species food chain system with toxicant and without toxicant with the help of numerical examples.

### 5.1 NUMERICAL EXAMPLE FOR MODEL 2

We choose the following values of parameters for $E_{1}^{*}$ :
$r_{0}=0.41 ; c=0.1 ; A_{1}=1.5 ; B_{1}=43.1 ; \beta_{10}=0.6 ; D_{1}=0.9$;
$B=153.3 ; a=0.5 ; A_{2}=0.5 ; B_{2}=8.9 ; \beta_{20}=0.1 ; D_{2}=0.01$.
It is found that under the above set of parameters, the equilibrium point $E_{1}{ }^{*}(12.5706,0,0)$ is locally asymptotically stable (see Fig.1).


Figure 1: Time graph for the Model 2 (without toxicant) around the equilibrium point $E_{1}^{*}$, showing the stability behavior.

We choose the following values of parameters for $E_{2}^{*}$ : $B=130.3 ; \quad A_{1}=0.8 ; \quad B_{1}=5.1 ; \beta_{10}=0.003$;
$D_{1}=0.001 ; A_{2}=0.003 ; B_{2}=6.5 ; \beta_{20}=0.02$.
With the above values of parameters and taking the remaining parameters to be the same as considered for $E_{1}^{*}$, it is found under the above set of parameters that the equilibrium $\quad \mathrm{E}_{2}{ }^{*} \quad(3.6744,1.225,0) \quad$ is locally asymptotically stable (see Fig.2).


Figure 2: Time graph for the Model 2 (without toxicant) around the equilibrium point $E_{2}^{*}$, showing the stability behavior.

We choose the following values of parameters for $\mathrm{E}_{3}^{*}$ :
$\mathrm{D} 1=0.001, \mathrm{~A} 1=1.35, \mathrm{~A} 2=0.501, \mathrm{~B} 2=12.0$.
With the above values of parameters and taking the remaining parameters to be the same as considered for $\mathrm{E}_{1}^{*}$, it is found that the interior equilibrium $\mathrm{E} 3^{*}=(9.5375$, $2.2163,1.3688$ ) is locally asymptotically stable (see Figs. 3 and 4).


Figure 3: Time graph for the Model 2 (without toxicant) around the equilibrium point $E_{3}^{*}$, showing the stability behavior


Figure 4: Phase graph for the Model 2 (without toxicant) around the equilibrium point $E_{3}^{*}$, showing the stability behavior.


Figure 5: Time graph for the Model 2 around the equilibrium point $E_{3}^{*}$, showing the bifurcation behavior.

Now, we study the Hopf-bifurcation of the Model 2, taking $A_{1}$ as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_{1}=A_{1}^{*}=0.6377$. It is clear that the interior equilibrium point $E_{3}^{*}$ of Model 2 is stable when $A_{1}>A_{1}^{*}$ and unstable when $A_{1} \leq A_{1}^{*}$ for which Hopfbifurcation occurs (see Figs. 5 and 6).


Figure 6: Phase graph for the Model 2 around the equilibrium point $E_{3}^{*}$, showing the bifurcation behavior.

### 5.2 NUMERICAL EXAMPLE FOR MODEL 1

To explain the applicability of the results discussed above, we consider the following particular forms of $r\left(C_{0}\right), \beta_{1}\left(C_{0}\right)$ and $\beta_{2}\left(C_{0}\right)$ in two cases as follows:

## Case One:

$r\left(C_{0}\right)=r_{0}-r_{1} C_{0}, \beta_{1}\left(C_{0}\right)=\beta_{10}-\beta_{11} C_{0}$ \&
$\beta_{2}\left(C_{0}\right)=\beta_{20}-\beta_{22} C_{0}$
We choose the following values of parameters for $E_{1 w}^{*}$ :
$r_{1}=0.081, \beta_{11}=0.5, \beta_{22}=0.031, a_{1}=2.8, d_{1}=0.01, \beta=0.8$, $\theta=0.31,1_{1}=1.2,1_{2}=0.55, \mathrm{~g}_{0}=0.2, \mathrm{k}_{1}=0.1, \mathrm{k}_{2}=0.12, \mathrm{r}_{0}=0.41$, $B=153.3, \quad c=0.1, \quad a=0.5, A_{1}=1.5, \quad B_{1}=43.1, \quad \beta_{10}=0.6$, $\mathrm{D}_{1}=0.9, \mathrm{~A}_{2}=0.5, \mathrm{~B}_{2}=8.9, \beta_{20}=0.1, \mathrm{D}_{2}=0.01$.
It is found that under the above set of parameters, the equilibrium point $\mathrm{E}_{1 \mathrm{w}}{ }^{*}$ (11.8853,0.0000,0.0000, $0.1723,0.2758$ ) is locally asymptotically stable (see Fig.7).
Now, we choose the following values of parameters for $E_{2 w}^{*}$ :


Figure 7: Time graph for the Model 1, Case One (with toxicant) around the equilibrium point $E_{1 w}^{*}$, showing the stability behavior.

With the above values of parameters and taking the remaining parameters to be the same as considered for $E_{1 w}^{*}$ of Model 1 (Case One), it is found that the equilibrium $\quad E_{2 w}{ }^{*} \quad(2.7181,0.9084,0,0.6886,0.7867)$ is locally asymptotically stable (see Fig.8).


Figure 8: Time graph for the Model 1, Case One (with toxicant) around the equilibrium point $E_{2 w}^{*}$, showing the stability behavior.

Now, we choose the following values of parameters for $E_{3 w}^{*}$ :

$$
\begin{aligned}
\mathrm{r}_{1}=0.2 ; \beta_{11}=0.08 ; \mathrm{a}_{1}=1.00 ; \mathrm{A}_{1}=1.1 ; \\
\mathrm{D}_{1}=0.001 ; \mathrm{B}_{2}=9.0 ; \mathrm{A}_{2}=0.6 .
\end{aligned}
$$

With the above values of parameters and taking the remaining parameters to be the same as considered for
$E_{1 w}^{*}$ of Model 1 (Case One), it is found that the interior equilibrium $E_{3 w}{ }^{*}(8.1611,2.0321,0.9378,0.535,0.3070)$ is locally asymptotically stable (see Figs. 9 and 10).


Figure 9: Time graph for the Model 1, Case One (with toxicant) around the equilibrium point $E_{3 w}^{*}$, showing the stability behavior


Figure 10: Phase graph for the Model 1, Case One (with toxicant) around the equilibrium point $E_{3 w}^{*}$, showing the stability behavior.


Figure 11: Time graph for the Model 1 (Case One) around the equilibrium point $E_{3 w}^{*}$, showing the bifurcation behavior.

Now, we study the Hopf-bifurcation of the Model 1, taking $A_{1}$ as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_{1}=A_{1}^{*}=0.7015$. It is clear that the interior equilibrium point $E_{3 w}^{*}$ of Model 1 is stable when
$A_{1}>A_{1}^{*}$ and unstable when $A_{1} \leq A_{1}^{*}$ for which Hopfbifurcation occurs (see Figs. 11 and 12).


Figure 12: Phase graph for the Model 1 (Case One) around the equilibrium point $E_{3 w}^{*}$, showing the bifurcation behavior

## Case Two:

$$
\begin{aligned}
& r\left(C_{0}\right)=r_{0}-r_{1} C_{0}, \beta_{1}\left(C_{0}\right)=\beta_{10} /\left(1+\beta_{11} C_{0}\right), \\
& \beta_{2}\left(C_{0}\right)=\beta_{20} /\left(1+\beta_{22} C_{0}\right)
\end{aligned}
$$

We choose the following values of parameters for $E_{1 w}^{*}$ : All the parameters to be the same as considered for $E_{1 w}^{*}$ of Model 1 (Case One), it is found that the equilibrium $E_{l w}{ }^{*} \quad(11.8853,0,0,0.1722,0.2759)$ is locally asymptotically stable (see Fig.13).


Figure 13: Time graph for the Model 1, Case Two (with toxicant) around the equilibrium point $E_{1 w}^{*}$, showing the stability behavior.

Now, we choose the following values of parameters for $E_{2 w}^{*}$ : $\mathrm{r}_{1}=0.03, \beta_{11}=0.5, \beta_{22}=0.021, \mathrm{a}_{1}=1.60, \mathrm{~g}_{0}=0.18, \mathrm{~A} 1=1.5$. With the above values of parameters and taking the remaining parameters to be the same as considered for $E_{2 w}^{*}$ of Model 1 (Case One), it is found that the equilibrium $\quad E_{2 w}{ }^{*} \quad(2.2030,0.8456,0,0.7809,0.7155)$ is locally asymptotically stable (see Fig.14).


Figure 14: Time graph for the Model 1, Case Two (with toxicant) around the equilibrium point $E_{2 w}^{*}$, showing the stability behavior. Now, for $E_{3 w}^{*}$, all the parameters to be the same as considered for $E_{3 w}^{*}$ of Model 1 (Case One), it is found that the interior equilibrium E3w* ( $8.4443,1.8201,0.9556,0.5226,0.3002$ ) is locally asymptotically stable (see Figs. 15 and 16).


Figure 15: Time graph for the Model 1, Case Two (with toxicant) around the equilibrium point $E_{3 w}^{*}$, showing the stability behavior.


Figure 16: Phase graph for the Model 1, Case Two (with toxicant) around the equilibrium point $E_{3 w}^{*}$, showing the stability behavior.


Figure 17: Time graph for the Model 1 (Case Two) around the equilibrium point $E_{3 w}^{*}$, showing the bifurcation behavior.

Now, we study the Hopf-bifurcation of the Model 1, taking $A_{1}$ as the bifurcating parameter. The transversality condition holds with the above set of parameters when $A_{1}=A_{1}^{*}=0.753$. It is clear that the interior equilibrium point $E_{3 w}^{*}$ of Model 1 is stable when $A_{1}>A_{1}^{*}$ and unstable when $A_{1} \leq A_{1}^{*}$ for which Hopfbifurcation occurs (see Figs. 17 and 18).


Figure 18: Phase graph for the Model 1 (Case Two) around the equilibrium point $E_{3 w}^{*}$, showing the bifurcation behavior.

### 5.3 EFFECT OF TOXICANT ON MODEL 1 AND COMPARISON WITH MODEL 2:

Now, we compare the equilibrium levels of the population for both the models. From the Table 1 to Table 3 and figures (19-24), we can see that the populations are decreasing under the stress of toxicant.

Table 1: Numerical values of equilibrium points of Model 2

| EquilibriumPoints | Numerical values of Model 2 |
| :--- | :--- |
| $E_{1}^{*}(\tilde{x}, 0,0)$ | $(12.571,0.000,0.000)$ |
| $E_{2}^{*}(\hat{x}, \hat{y}, 0)$ | $(3.6744,1.2250,0.000)$ |
| $E_{3}^{*}\left(x^{*}, y^{*}, z^{*}\right)$ | $(9.537,2.216,1.3688)$ |

Table 2: Numerical values of equilibrium points of Model 1, Case One

| Equilibrium Points | Numerical values of <br> Model 1, Case One |
| :--- | :---: |
| $E_{1 w}^{*}\left(\tilde{x}, 0,0, \tilde{C}_{0}, \tilde{C}_{E}\right)$ | $(11.8853,0,0$, |
| $0.1723,0.2758)$ |  |
| $E_{2 w}^{*}\left(\hat{x}, \hat{y}, 0, \hat{C}_{0}, \hat{C_{E}}\right)$ | $(2.7181,0.9084,0$, |
| $E_{3 w}^{*}\left(x^{*}, y^{*}, z^{*}, C_{0}^{*}, C_{E}^{*}\right)$ | $(8.6886,0.7867)$ |



Figure 19: Time graph for basal resource population of Model 2 compared with Model 1 (Case One) around the equilibrium points $E_{1}$ and $E_{1 w}^{*}$ respectively, showing the stability behavior.


Figure 20: Time graph for basal resource population of Model 2 compared with Model 1 (Case Two) around the equilibrium points $E_{1}^{*}$ and $E_{1 w}^{*}$ respectively, showing the stability behavior.

Table 3: Numerical values of equilibrium points of Model 1, Case Two

| Equilibrium Points | Numerical values of <br> Model 1, Case Two |
| :--- | :--- |
| $E_{1 w}^{*}\left(\tilde{x}, 0,0, \tilde{C}_{0}, \tilde{C}_{E}\right)$ | $(11.885,0.000$, |
|  | $0.000,0.1722,0.2759)$ |
| $E_{2 w}^{*}\left(\hat{x}, \hat{y}, 0, \hat{C}_{0}, \hat{C}_{E}\right)$ | $(2.203,0.8456,0.000$, <br> $0.7809,0.7155)$ |
| $E_{3 w}^{*}\left(x^{*}, y^{*}, z^{*}, C_{0}^{*}, C_{E}^{*}\right)$ | $(8.444,1.820,0.955$, |



Figure 21: Time graph for basal resource and intermediate consumer populations of Model 2 compared with Model 1 (Case One) around the equilibrium points $E_{2}^{*}$ and $E_{2 w}^{*}$ respectively, showing the stability behavior.


Figure 22: Time graph for basal resource and intermediate consumer populations of Model 2 compared with Model 1 (Case Two) around the equilibrium points $E_{2}^{*}$ and $E_{2 w}^{*}$ respectively, showing the stability behavior.


Figure 23: Time graph for Model 2 compared with Model 1 (Case One) around the equilibrium points $E_{3}^{*}$ and $E_{3 w}^{*}$ respectively, showing the stability behavior.


Figure 24: Time graph for Model 2 compared with Model 1 (Case Two) around the equilibrium points $E_{3}^{*}$ and $E_{3 w}^{*}$ respectively, showing the stability behavior.

## VI. CONCLUSION

In this paper we have proposed and analyzed a nonlinear mathematical model to study the effect of toxicant on a three species food chain system. The local stability analysis of all the equilibrium points of the Model 1 and 2 has been carried out. The global stability analysis of only the non-trivial positive equilibrium points of both the Models has been conducted. From the stability of $E_{1}$ of Model 2, it is concluded that only the prey population will survive and both the predator populations would tend to extinction. From the stability of $E_{1 w}^{*}$ of Model 1 we derive the same dynamical behavior of prey and predator populations as observed for $E_{1}^{*}$ of Model 2 with the only difference that equilibrium level of prey population reduces due to the presence of toxicant (see

Figs.1, 7, 13, 19 and 20). From the stability of $E_{2}^{*}$ of Model 2, it is concluded that only prey and intermediate predator populations would survive and the top predator population may die out. Similar dynamical behavior has been observed for prey and predator populations from the stability analysis of $E_{2 w}^{*}$ as being observed from the stability analysis of $E_{2}^{*}$. However, in this case also the equilibrium level of prey and predator populations decrease due to the presence of toxicant (see Figs.2, 8, 14, 21 and 22). The interior equilibrium points of both the Models are locally stable showing the same dynamical behavior and co-existence of all the three populations of prey and predator species. However, from the equilibrium values it is seen that the equilibrium density of top predator reduces due to the presence of toxicant in prey and intermediate predator. It may be also noted from the equilibrium of the intermediate predator population that the level of intermediate predator population may increase due to the presence of toxicant in the top predator.

The interior equilibrium points of both the Models are globally asymptotically stable in the regions $\Omega_{1}$ and $\Omega_{2}$. Looking at $\Omega_{1}$ and $\Omega_{2}$, it may be concluded that the region of global stability shrinks when the toxicant is introduced in the underlying system of prey and predator species. It is noted from the stability conditions of the equilibrium of the models that the system with toxicant seems to be more stable than that of the system with no toxicant effects. It is further concluded that the system with toxicant moves faster towards equilibrium after given perturbation than that of the system without toxicant for same parametric values. Finally, we have demonstrated the dynamical behavior of a three species food chain system with toxicant and without toxicant with the help of numerical simulation to support analytical results.

## VII. APPENDIX

## APPENDIX A (Proof of Theorem 3.2)

We consider the following positive definite function about $E_{3}^{*}$ :
$V=\left(x-x^{*}-x^{*} \ln \left(x / x^{*}\right)\right)+(1 / 2)\left(y-y^{*}\right)^{2}+\left(G_{1} / 2\right)\left(z-z^{*}\right)^{2}$

Differentiating $V$ with respect to time $t$, we get $\dot{V}=\left(x-x^{*} / x\right)(d x / d t)+\left(y-y^{*}\right)(d y / d t)+G_{1}\left(z-z^{*}\right)(d z / d t)$ Using system of equations (11)-(13), we get after some algebraic manipulations
$\dot{V}=-\left(x-x^{*}\right)^{2}\left[\frac{a T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a x\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}\right]$
$-\left(y-y^{*}\right)^{2}\left[\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{10} x^{*}}{(1+z)\left(B_{1}+x^{*}\right)}\right]$
$-\left(z-z^{*}\right)^{2} G_{1}\left[D_{2}-\frac{A_{2} \beta_{20} y^{*}}{B_{2}+y^{*}}\right]$
$+\left(x-x^{*}\right)\left(y-y^{*}\right)\left[\frac{A_{1} B_{1} \beta_{10} y}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}-\frac{A_{1}}{(1+z)\left(B_{1}+x\right)}\right]$
$+\left(x-x^{*}\right)\left(z-z^{*}\right)\left[\frac{A_{1} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right]$
$\left.+\left(y-y^{*}\right)\left(z-z^{*}\right)\left[\frac{A_{2} B_{2} G_{1} \beta_{20} z}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)-\left(\frac{A_{1} \beta_{10} 0^{*} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right.}+\frac{A_{2} y}{B_{2}+y}\right)\right]$
Now, $\dot{V}$ can further be written as sum of the quadratic forms as
$\dot{V} \leq-\left[\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{12}\left(x-x^{*}\right)\left(y-y^{*}\right)+\left(a_{22} / 2\right)\left(y-y^{*}\right)^{2}\right)\right.$
$+\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{13}\left(x-x^{*}\right)\left(z-z^{*}\right)+\left(a_{33} / 2\right)\left(z-z^{*}\right)^{2}\right)$
$\left.+\left(\left(a_{22} / 2\right)\left(z-z^{*}\right)^{2}-a_{23}\left(y-y^{*}\right)\left(z-z^{*}\right)+\left(a_{33} / 2\right)\left(z-z^{*}\right)^{2}\right)\right]$
where,
$a_{11}=\left(\frac{a T_{0}\left(c+r_{0}\right)}{\left(T_{0}+a x\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}\right)$,
$a_{12}=\left(\frac{A_{1} B_{1} \beta_{10} y}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}-\frac{A_{1}}{(1+z)\left(B_{1}+x\right)}\right)$,
$a_{13}=\left(\frac{A_{1} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right.$,
$a_{22}=\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{10} x^{*}}{(1+z)\left(B_{1}+x^{*}\right)}\right)$,
$a_{23}=\left(\frac{A_{2} B_{2} G_{1} \beta_{20} z}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}-\left(\frac{A_{1} \beta_{10} x^{*} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}+\frac{A_{z} y}{B_{2}+y^{2}}\right)\right.$,
$a_{33}=G_{1}\left(D_{2}-\frac{A_{2} \beta_{20} y^{*}}{B_{2}+y^{*}}\right)$,
Sufficient conditions for $\dot{V}$ to be negative definite are that the following inequalities hold:
$a_{11}>0$
$a_{22}>0$
$a_{11} a_{22}>a_{12}^{2}$
$a_{11} a_{33}>a_{13}^{2}$
$a_{22} a_{33}>a_{23}^{2}$
We note that the fourth inequality, i.e., $a_{11} a_{33}>a_{13}^{2}$ is satisfied due to the proper choice of $G_{1},(34) \Rightarrow(61)$, $(35) \Rightarrow(62),(36) \Rightarrow(63)$ and $(37) \Rightarrow(65)$. Hence V is a Lyapunov function with respect to $E_{3}^{*}$, whose domain contains the region of attraction $\Omega_{2}$, proving the theorem.

## APPENDIX B (Proof of Theorem 4.2)

We consider the following positive definite function about $E_{3 w}^{*}$ :
$V=\left(x-x^{*}-x^{*} \ln \left(x / x^{*}\right)\right)+\left(E_{1} / 2\right)\left(y-y^{*}\right)^{2} \quad+\left(E_{2} / 2\right)(z-$
$\left.z^{*}\right)^{2}+\left(E_{3} / 2\right)\left(C_{0}-C_{0}{ }^{*}\right)^{2}+\left(E_{4} / 2\right)\left(C_{E}-C_{E}{ }^{*}\right)^{2}$
Differentiating $V$ with respect to time $t$, we get
$\dot{V}=\left(x-x^{*}\right) / x(d x / d t)+E_{l}\left(y-y^{*}\right)(d y / d t)+E_{2}(z-$
$\left.z^{*}\right)(d z / d t)+E_{3}\left(C_{0^{-}} C_{0}{ }^{*}\right)\left(d C_{0} d t\right)+E_{4}\left(C_{E^{-}}\right.$
$\left.C_{E}{ }^{*}\right)\left(d C_{E} / d t\right)$
Using system of equations (1)-(5), we get after some algebraic manipulations
$\dot{V}=-\left(x-x^{*}\right)^{2}\left(\frac{a T_{0}\left(c+r\left(C_{0}^{*}\right)\right)}{\left(T_{0}+a x\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}\right)$
$-\left(y-y^{*}\right)^{2} E_{1}\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*}}{(1+z)\left(B_{1}+x^{*}\right)}\right)$
$-\left(z-z^{*}\right)^{2} E_{2}\left(D_{2}-\frac{A_{2} \beta_{2}\left(C_{0}^{*}\right) y^{*}}{B_{2}+y^{*}}\right)-\left(C_{0}-C_{0}^{*}\right)^{2} E_{3}\left(l_{1}+l_{2}\right)$
$-\left(C_{E}-C_{E}^{*}\right)^{2} E_{4}\left(k_{1} a_{1} x^{*}+k_{2}\right)$
$+\left(x-x^{*}\right)\left(y-y^{* *}\right)\left(\frac{A_{1} B_{1} E_{1} \beta_{1}\left(C_{0}^{*}\right) y}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}-\frac{A_{1}}{(1+z)\left(B_{1}+x\right)}\right)$
$+\left(x-x^{*}\right)\left(z-z^{*}\right)\left(\frac{A y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}\right)$
$+\left(x-x^{*}\right)\left(C_{0}-C_{0}^{*}\right)\left(\frac{T_{0} \delta\left(C_{0}\right)}{T_{0}+a x}\right)$
$+\left(x-x^{*}\right)\left(C_{E}-C_{E}^{*}\right)\left(E_{4} k_{1}\left(l_{1} C_{0}^{*}-a_{1} C_{E}\right)\right)$
$+\left(y-y^{*}\right)\left(z-z^{*}\right)\left(\frac{A_{2} B_{2} E_{2} \beta_{2}\left(C_{0}^{*}\right) z}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}\right.$
$-E_{1}\left(\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)^{*}}+\frac{A_{2} y}{B_{2}+y}\right)$
$+\left(y-y^{*}\right)\left(C_{0}-C_{0}^{*}\right)\left(\frac{A_{1} E_{1} \alpha\left(C_{0}\right) x y}{(1+z)\left(B_{1}+x\right)}\right)$
$+\left(z-z^{*}\right)\left(C_{0}-C_{0}^{*}\right)\left(\frac{A_{2} E_{2} \gamma\left(C_{0}\right) y z}{B_{2}+y}\right)$
$+\left(C_{0}-C_{0}^{*}\right)\left(C_{E}-C_{E}^{*}\right)\left(a_{1} E_{3}+E_{4} k_{1} l_{1} x\right)$
where,
$\delta\left(C_{0}\right)=\left\{\begin{array}{cc}\frac{r\left(C_{0}\right)-r\left(C_{0}^{*}\right)}{C_{0}-C_{0}^{*}} ; C_{0} \neq C_{0}^{*} \\ \dot{r}\left(C_{0}^{*}\right) ; & C_{0}=C_{0}^{*}\end{array}\right.$
$\alpha\left(C_{0}\right)=\left\{\begin{array}{cc}\frac{\beta_{1}\left(C_{0}\right)-\beta_{1}\left(C_{0}^{*}\right)}{C_{0}-C_{0}^{*}} ; & C_{0} \neq C_{0}^{*} \\ \beta_{1}\left(C_{0}^{*}\right) ; & C_{0}=C_{0}^{*}\end{array}\right.$
$\gamma\left(C_{0}\right)=\left\{\begin{array}{cc}\frac{\beta_{2}\left(C_{0}\right)-\beta_{2}\left(C_{0}^{*}\right)}{C_{0}-C_{0}^{*}} ; & C_{0} \neq C_{0}^{*} \\ \beta_{2}^{*}\left(C_{0}^{*}\right) ; & C_{0}=C_{0}^{*}\end{array}\right.$
Now, $\dot{V}$ can further be written as sum of the quadratic forms:

$$
\begin{aligned}
& \dot{V} \leq-\left[\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{12}\left(x-x^{*}\right)\left(y-y^{*}\right)+\left(a_{22} / 2\right)\left(y-y^{*}\right)^{2}\right)\right. \\
& +\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{13}\left(x-x^{*}\right)\left(z-z^{*}\right)+\left(a_{33} / 2\right)\left(z-z^{*}\right)^{2}\right) \\
& +\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{14}\left(x-x^{*}\right)\left(C_{0}-C_{0}^{*}\right)+\left(a_{44} / 2\right)\left(C_{0}-C_{0}^{*}\right)^{2}\right) \\
& +\left(\left(a_{11} / 2\right)\left(x-x^{*}\right)^{2}-a_{15}\left(x-x^{*}\right)\left(C_{0}-C_{0}^{*}\right)+\left(a_{55} / 2\right)\left(C_{E}-C_{E}^{*}\right)^{2}\right) \\
& +\left(\left(a_{22} / 2\right)\left(y-y^{*}\right)^{2}-a_{23}\left(y-y^{*}\right)\left(z-z^{*}\right)+\left(a_{33} / 2\right)\left(z-z^{*}\right)^{2}\right) \\
& +\left(\left(a_{\left.\left.22^{\prime} / 2\right)\left(y-y^{*}\right)^{2}-a_{24}\left(y-y^{*}\right)\left(C_{0}-C_{0}^{*}\right)+\left(a_{44} / 2\right)\left(C_{0}-C_{0}^{*}\right)^{2}\right)}^{+\left(\left(a_{33} / 2\right)\left(z-z^{*}\right)^{2}-a_{34}\left(z-z^{*}\right)\left(C_{0}-C_{0}^{*}\right)+\left(a_{44} / 2\right)\left(C_{0}-C_{0}^{*}\right)^{2}\right)}\right.\right. \\
& \left.+\left(\left(a_{44} / 2\right)\left(C_{0}-C_{0}^{*}\right)^{2}-a_{45}\left(C_{0}-C_{0}^{*}\right)\left(C_{E}-C_{E}^{*}\right)+\left(a_{55} / 2\right)\left(C_{E}-C_{E}^{*}\right)^{2}\right)\right]
\end{aligned}
$$

where,
$a_{11}=\frac{1}{2}\left(\frac{a T_{0}\left(c+r\left(C_{0}^{*}\right)\right)}{\left(T_{0}+a x\right)\left(T_{0}+a x^{*}\right)}-\frac{A_{1} y^{*}}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}\right)$,
$a_{12}=\frac{A_{1} B_{1} E_{1} \beta_{1}\left(C_{0}^{*}\right) y}{(1+z)\left(B_{1}+x\right)\left(B_{1}+x^{*}\right)}-\frac{A_{1}}{(1+z)\left(B_{1}+x\right)}$,
$a_{13}=\frac{A_{1} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}, a_{14}=\frac{T_{0} \delta\left(C_{0}\right)}{T_{0}+a x}$,
$a_{15}=E_{4} k_{1}\left(l_{1} C_{0}^{*}-a_{1} C_{E}\right)$,
$a_{22}=\frac{2 E_{1}}{3}\left(\frac{A_{2} B_{2} z^{*}}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}+D_{1}-\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*}}{(1+z)\left(B_{1}+x^{*}\right)}\right)$,
$a_{23}=\frac{A_{2} B_{2} E_{2} \beta_{2}\left(C_{0}^{*}\right) z}{\left(B_{2}+y\right)\left(B_{2}+y^{*}\right)}-E_{1}\left(\frac{A_{1} \beta_{1}\left(C_{0}^{*}\right) x^{*} y^{*}}{(1+z)\left(1+z^{*}\right)\left(B_{1}+x^{*}\right)}+\frac{A_{2} y}{B_{2}+y}\right)$,
$a_{24}=\frac{A_{1} E_{1} \alpha\left(C_{0}\right) x y}{(1+z)\left(B_{1}+x\right)}, a_{33}=\frac{2 E_{2}}{3}\left(D_{2}-\frac{A_{2} \beta_{2}\left(C_{0}^{*}\right) y^{*}}{B_{2}+y^{*}}\right)$
$, a_{34}=\left[\frac{A_{2} E_{2} \gamma\left(C_{0}\right) y z}{B_{2}+y}\right], a_{44}=\frac{E_{3}}{2}\left(l_{1}+l_{2}\right)$,
$a_{45}=a_{1} E_{3}+E_{4} k_{1} l_{1} x, a_{55}=E_{4}\left[k_{1} a_{1} x^{*}+k_{2}\right]$.
Sufficient conditions for $\dot{V}$ to be negative definite are that the following inequalities hold:

$$
\begin{align*}
& a_{11}>0  \tag{66}\\
& a_{22}>0  \tag{67}\\
& a_{11} a_{22}>a_{12}^{2}  \tag{68}\\
& a_{11} a_{33}>a_{13}^{2}  \tag{69}\\
& a_{11} a_{44}>a_{14}^{2} \tag{70}
\end{align*}
$$

$a_{11} a_{55}>a_{15}^{2}$
$a_{22} a_{33}>a_{23}^{2}$
$a_{22} a_{44}>a_{24}^{2}$
$a_{33} a_{44}>a_{34}^{2}$
(74) $a_{44} a_{55}>a_{45}^{2}$

We note that the fourth, fifth, sixth, eighth and ninth inequalities, i.e., $a_{11} a_{33}>a_{13}^{2}, a_{11} a_{44}>a_{14}^{2}, a_{11} a_{55}>a_{15}^{2}$, $a_{22} a_{44}>a_{24}^{2}$ and $a_{33} a_{44}>a_{34}^{2}$ are satisfied due to the proper choice of $E_{1}, E_{2}, E_{3}, E_{4}$ and other inequalities, $(52) \Rightarrow(66), \quad(53) \Rightarrow(67), \quad(54) \Rightarrow(68), \quad(55) \Rightarrow(72) \quad$ and $(56) \Rightarrow(75)$. Hence $V$ is a Lyapunov function with respect to $E_{3 w}^{*}$, whose domain contains the region of attraction $\Omega_{1}$, proving the theorem.

## VIII. REFERENCES

[1]. A.A. Gomes, E. Manica and M.C. Varriale, 2008, Applications of chaos control techniques to a threespecies food chain, Chaos, Solitons and Fractals 36: 1097-1107.
[2]. Alan Hastings and Thosmas Powell, 1991, Chaos in a Three-species food chain, Ecology 72(3) : 896-903.
[3]. Chengjun Sun and Michel Loreau, 2009, Dynamics of a three-species food chain model with adaptive traits, Chaos, Solitons and Fractals 41: 2812-2819.
[4]. De Luna J.T. and Hallam T.G., 1987, Effect of toxicants on population: a qualitative approach IV. Resource Consumer - Toxicant models, Ecol Modelling 35: 249273.
[5]. Debaldev Jana, Rashmi Agrawal, Ranjit Kumar Upadhyay, 2014, Top-predator interference and gestation delay as determinants of the dynamics of a realistic model food chain, Chaos, Solitons and Fractals 69:50-63.
[6]. Fengyan Wang and Guoping Pang, 2008, Chaos and Hopf bifurcation of a hybrid ratio-dependent three species food chain, Chaos, Solitons and Fractals 36: 1366-1376.
[7]. Gakkhar S and Naji M.A., 2003, Order and Chaos in a predator to prey ratio-dependent food chain, Chaos, Solitons and Fractals 18: 229-239.
[8]. Gakkhar S and Singh B., 2006. Dynamics of a modified Leslie-Gower-type prey-predator models with seasonally varying parameters, Chaos, Solitons \& Fractals 27: 12391255.
[9]. Gakkhar S, Singh B and Naji R.K., 2007, Dynamical behavior of two predators competing over a single prey, Bio Systems 90:808-817.
[10]. George A.K. van Voorn, Bob W. Kooi and Martin P. Boer., 2010, Ecological consequences of global bifurcations in some food chain models, Mathematical Biosciences 226:120-133.
[11]. H.I. Freedman and J.B. Shukla., 1991, Models for the effect of toxicant in single-species and predator-prey systems, Jornal of Mathematical Biology 30:15-30.
[12]. H. I. Freedman and Paul Waltman, 1977, Mathematical Analysis of Some Three-Species Food-Chain Models, Mathematical Biosciences 33:257-276.
[13]. Huitao Zhao, Yiping Lin and Yunxian Dai, 2011, Bifurcation analysis and control of chaos for a hybrid ratio-dependent three species food chain, Applied Mathematics and Computation 218:1533-1546.
[14]. J.B. Shukla and B. Dubey, 1996, Simultaneous effect of two toxicants on biological species: A Mathematical Model, Journal of Biological Systems 4:109-130.
[15]. J.B. Shukla, A.K. Agrawal, B. Dubey and P.Sinha, 2001, Existence and Survival of Two Competing Species in a Polluted Environment: A Mathematical Model, Journal Biological Systems 9:89-103.
[16]. Kejun Zhuang and Zhaohui Wen, 2011, Dynamics of a Discrete Three Species Food Chain System, Int. J. of Comp. \& Math. Sci. 5.
[17]. Liu, W.M., 1994, Criterion of Hopf-bifurcations without using eigenvalues, J. Math. Anal. Appl. 182: 250-256.
[18]. Mada Sanjaya Waryano Sunaryo, Mustafa Mamat, Zabidin Salleh, Ismail Mohd and Noor Maizura Mohamad Noor, 2011, Numerical Simulation Dynamical Model of Three Species Food Chain with Holling TypeII Functional Response, Malaysian Journal of Mathematical Sciences 5: 1-12.
[19]. Mainul Haque, Nijamuddin Ali, Santabrata Chakravarty, 2013, Study of a tri-trophic prey-dependent food chain model of interacting populations, Mathematical Biosciences 246: 55-71.
[20]. Manju Agarwal and Sapna Devi, 2011, A resourcedependent competition model: Effects of toxicants emitted from external sources as well as formed by precursors of competing species, Nonlinear Analysis: Real World Applications 12:751-766.
[21]. Nitu Kumari, 2013, Pattern Formation in Spatially Extended Tritrophic Food Chain Model Systems: Generalist versus Specialist Top Predator, Hindawi Publishing Corporation, ISRN Biomathematics,Article ID 198185, 12 pages.
[22]. O.P.Misra and P.Sinha, 2007, Effect of Pollution on the Photosynthate Partitioning during Plant Growth: A three Compartment Model, Research Hunt 2.
[23]. O.P. Misra and Raveendra Babu.A, 2014, A model for the effect of toxicant on a three species food-chain system with "food-limited" growth of prey population, Global Journal of Mathematical Analysis, 2 (3) :120145.
[24]. O.P. Misra and V.P. Saxena, 1991, Effects of environmental pollution on the growth and existence of biological populations: Modelling and stability analysis, Indian J. pure appl. Math. 22: 805-815.
[25]. Qihua Huang, Laura Parshotam, HaoWang, Caroline Bampfylde and Mark A. Lewis, 2013, A model for the impact of contaminants on fish population dynamics, Journal of Theoretical Biology 334: 71-79.
[26]. R.K. Naji and A.T. Balasim, 2007, On the dynamical behavior of three species food web model, Chaos, Solitons and Fractals 34 :1636-1648.
[27]. Raid Kamel Naji, Ranjit Kumar Upadhyay and Vikas Rai, 2010, Dynamical consequences of predator interference in a tri-trophic model food chain, Nonlinear Analysis: Real World Applications 11: 809-818.
[28]. Robert V. Thomann, Daniel S. Szumski, Dominic M.Ditoto and Donald J O'Connor, 1984, A food chain model of cadmium in western lake Erie, Wat. Res. 8: 841-849.
[29]. Robert V. Thomann and John P. Connolly, 1984, Model of PCB in the Lake Michigan lake trout these food chain, Environ. Sci. Technol. 18:65-71.
[30]. Songjuan Lv and Min Zhao, 2008, The dynamic complexity of a three species food chain model, Chaos, Solitons and Fractals 37:1469-1480.
[31]. Steven J. Hamilton, 2004, Review of selenium toxicity in the aquatic food chain, Science of the Total Environment 326: 1-31.
[32]. Sudipa Sinha, O.P. Misra and J. Dhar, 2010, Modelling a Predator-Prey System with Infected Prey in Polluted Environment, Applied Mathematical Modelling 34:1861-1872.
[33]. Sudipa Sinha, O.P. Misra and Joydip Dhar, 2010, A two species competition model under the simultaneous effect of toxicant and disease, Nonlinear Analysis: Real World Applications 11:1131-1142.
[34]. Swati Khare, O. P. Misra, Chhatrapal Singh and Joydip Dhar, 2011, Role of Delay on Planktonic Ecosystem in the Presence of a Toxic Producing Phytoplankton, Hindawi Publishing Corporation, International Journal of Differential Equations, ID 603183, 16 pages
[35]. T. Das, R.N. Mukherjee and K.S. Chaudhuri, 2009, Harvesting of a prey-predator fishery it the presence of toxicity, Appl. Math. Model. 33: 2282-2292.
[36]. T.G. Hallam and C.E. Clark, 1982, Non-autonomous logistic equations as models of populations in a deteriorating environment, J. Theor. Biol. 93:303-311.
[37]. T.G. Hallam and J.T. De. Luna, 1984, Effects of toxicants on Populations: a Qualitative Approach III. Environmental and Food Chain Pathways, Academic Press Inc. (London) Ltd.
[38]. T.G. Hallam, Clark C.E. and Lassiter R.R., 1983, Effects of toxicants on population: a qualitative approach I.

Equilibrium environmental exposure, Ecol Modelling 18 :291-304.
[39]. T.G. Hallam, Clark C.E. and Jordan G.S., 1983, Effects of toxicants on population: a qualitative approach II. First order kinetics, J. Math. Biol. 18:25-37.
[40]. Xitao Wang and Min Zhao., 2011, Chaos in a Hybrid Three-Species Food Chain with Beddington-Deangelis Functional Response, Procedia Environmental Sciences 10:128-134.
[41]. Zhao M and Lv S., 2009a, Chaos in a three-species food chain model with a Beddington-DeAngelis functional response, Chaos, Solitons and Fractals 40: 2305-2316.
[42]. Mini Ghosh, Peeyush Chandra and Prawal Sinha, 2002, A Mathematical Model To Study The Effect Of Toxic Chemicals On A Prey-Predator Type Fishery, Journal of Biological Systems, Vol. 10, No. 2: 97-105.
[43]. Graeme M. Smith, Judith S. Weis, 1997, Predator-prey relationships in mummichogs (Fundulus heteroclitus (L.)): Effects of living in a polluted environment, Journal of Experimental Marine Biology and Ecology, 209: 75-87.
[44]. Jes Jessen Rasmussena, Ulrik Norum, Morten Rygaard Jerris, Peter Wiberg-Larsen, Esben Astrup Kristensen, Nikolai Friberg, 2013, Pesticide impacts on predatorprey interactions across two levels of organisation, Aquatic Toxicology 140-141: 340-345.

