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Metric - A Review

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ABSTRACT

Some of the line elements in their original forms as they appear in literatures in different system of co-ordinates and as used by different workers have been presented in this work. Their characteristics and possible uses have also been discussed. The types of substitutions made by the workers to modify a metric for specific use have also been mentioned. The line elements have been divided, arbitrarily, into several groups for ease of discussion. The review has been made in brief as far as practicable.

Keywords : 3-volume, Static Weak Field, Null Tetrad, Oblate Spherical Co-Ordinates, Hyper Sphere, Laboratory Foliation Co-Frame.

I. INTRODUCTION

Frequently, we are in need of an interval (i.e. a line element or a metric) to study the events occurring in the universe at points infinitely closed in respect of space as well as time. What is a metric?

" An expression that expresses the separation ds between two infinitely adjacent points is called an interval, a line element or a metric."[1]

It represents space-time geometry and is expressed by $(ds)^2$. The characteristics of a metric have been discussed by Hartle [2], Puri [3], Ugarov [4] and others. Although, the types of expressions of metric found in literatures is too large, a few of them would be taken into consideration and be presented in this review type dissertation. Trial would be made to mention their characteristics and uses as and when possible. It could, also, be seen that the sign of the terms in the expression of line element differs as it is used by different workers. We shall retain them in the same form and same unit as they appear in literatures.

Some basic expressions

Some basic expressions commonly found in literatures have been presented here. Generally, in 2-D the intervals are expressed respectively in cartesian and in polar co-ordinates as shown below [2, 5, 6].

$$ds^2 = dx^2 + dy^2 \qquad (1)$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 \qquad (2)$$

A classic example of metric considering 2-D hyperbolic plane is given by [2]

$$ds^2 = y^{-2}(dx^2 + dy^2) \qquad (3)$$

Again, for 3-D the line element could be written as [1, 3]

$$ds^2 = dx^2 + dy^2 + dz^2 \qquad (4)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \qquad (5)$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \mathrm{Sin}^2\theta \, d\phi^2 \qquad (6)$$

respectively in cartesian,cylindrical and spherical polar co-ordinates.

In the simplest case if x, y, z, w be the cartesian co-ordinates of a point in 4-D Euclidean space then ds would be given by [5, 6]

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2 \qquad (7)$$

Now, we shall shift our attention to 4-D space-time due to Herman Minkowski (1907). Considering Minkowski space, we have [4]

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \qquad (8)$$

From (7) and (8) we see that the signs of co-efficients of the terms differ. Also, in special relativity the 3-D space co-ordinates and time co-ordinates combine in an invariant manner to have 4-D space-time.

Now, it is noteworthy that ds is invariant with respect to system of co-ordinates [2] but (8) could, also, be written as

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \quad (9)$$

Both (8) and (9) show that, ds^2 consists of the space and time parts which could be written as

$$ds^2 = c^2dt^2 - \sum(dx^i)^2 \quad (10)$$

where, i = 1, 2, 3 in the 3-D space. The two parts in (10) have different signs unlike that in 4-D Euclidean space-time as in (7). So, the 4-D space-time presented in (8) or (9) is called quasi-Euclidean flat space-time.

Transforming the spatial part of the metric (9) to spherical polar co-ordinates and for spherical symmetry (8) could be written as [2]

$$ds^2 = dt^2 - dr^2 - r^2d\theta^2 - r^2\text{Sin}^2\theta\, d\phi^2 \quad (11)$$

The characteristics of this equation are –

i) It is in relativistic unit,

ii) The isolated particle is assumed to be static and spherically symmetric,

iii) The gravitational field would depend upon r only but not on θ and φ,

iv) The line element would be spatially spherically symmetric about the point and itself is a static.

Again, (11) could also be written as

$$ds^2 = -dt^2 + dr^2 + r^2d\theta^2 + r^2\text{Sin}^2\theta\, d\phi^2 \quad (12)$$

Some common line elements

We know that a metric is radially symmetric. So, we may put co-efficients before each term of (11) which are functions of r only and arrive at

$$ds^2 = A\, dt^2 - B\, dr^2 - r^2(d\theta^2 + \text{Sin}^2\theta\, d\phi^2) \quad (13)$$

Similarly, Dingle [7] has used such co-efficients and obtained ds as [5]

$$ds^2 = -A(dx^1)^2 - B(dx^2)^2 - C(dx^3)^2 + D(dx^4)^2 \quad (14)$$

where, A, B, C and D are functions of x^1, x^2, x^3, x^4.

Now, in order to conform to the metric of (8) we have to preserve the signature of space. This is done by writing the co-efficients as exponentials since exponentials are always positive. Thus, (13) could be written in the most general form for static distribution of matter with spherical symmetry as [3]

$$ds^2 = e^{\nu}dt^2 - e^{\lambda}dr^2 - r^2d\theta^2 - r^2\text{Sin}^2\theta\, d\phi^2 \quad (15)$$

where, ν and λ are function of r only. It is to be mentioned that several authors have considered ν and λ to be functions of r and t to obtain the most general line element with spherical symmetry [5]. It would be of the same form as (15).

Now, (15) reveals that the gravitational field due to a particle decreases as we move away from it and (15) will become Galilean line element at r = ∞ making λ= 0 = ν.

Again, the physical events of general theory of relativity could be described in a 4-D space-time manifold by the tensor form as the metric [1, 3]

$$ds^2 = g_{ij}dx^idx^j \quad (16)$$

where, i, j = 1, 2, 3, 4 and g_{ij} are functions of the co-ordinates x. It is true in a curved space-time for an accelerated system or in a gravitational field. It is to be mentioned that (16) is in accordance with the principle of covariance and is invariant under Lorentz transformation. Of course, for a frame K' moving in a straight line with acceleration 'a' relative to another frame K the interval mentioned in (8) would become [4]

$$ds'^2 = (c^2 - a^2t'^2)dt'^2 - 2at'dx'dt' - dx'^2 - dy'^2 - dz'^2 \quad (17)$$

in K'. It is to note that there is a mixed term containing $dx'\,dt'$ due to the choice of a function.

Let us take the line element [2] in 3-D as an example

$$ds^2 = r^2\{d\theta^2 + f^2(\theta)d\phi^2\} \quad (18)$$

then it is evident that the form depends upon the choice of f(θ). For f(θ)= Sin θ we shall get the following line element defining the geometry of the non-Euclidean surface of a sphere of radius r [2, 3]

$$ds^2 = r^2(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (19)$$

Also, the metric of a sphere at the north pole [2] could be obtained from (6) by substituting $\theta = \frac{\sqrt{x^2+y^2}}{r}$ and $\phi = \tan^{-1}(\frac{y}{x})$

Now, for a sphere of radius R centred on r= 0 in this space the circumference around the equator, surface area, volume and distance from centre to surface of it could be calculated by using the spatial metric [2]

$$ds^2 = \frac{dr^2}{1-(\frac{r}{a})^2} + r^2(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (20)$$

Here, 'a' is a constant related to density of matter. Sometimes (20) is written as [2]

$$ds^2 = \frac{dr^2}{1-\frac{2M}{r}} + r^2(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (21)$$

From which the radial distance and the spatial volume could be calculated. Also, to find area and volume elements of a sphere [2] the metric to be used in flat space-time polar co-ordinates is (11) from which an element of area $dA = r^2 Sin\theta\, d\theta\, d\varphi$ and 3-volume $dv = r^2 Sin\theta\, dr\, d\theta\, d\varphi$ could be obtained.

Again, a slightly modified form of the metric used for finding the above parameters is [2]

$$ds^2 = -(1-Ar^2)^2 dt^2 + (1-Ar^2)^2 dr^2 + r^2(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (22)$$

It is to be mentioned that an example of a useful co-ordinate system for flat space is that which is used to construct Penrose diagram. Using (12) and replacing t and r by two new co-ordinates v (= t – r) and u (= t + r) we get the line element to be [2]

$$ds^2 = -du\, dv + \frac{1}{4}(u-v)^2(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (23)$$

Now, to introduce a slight curvature (i.e. metric for static weak field) the flat space-time geometry of special relativity is to be modified to [2]

$$ds^2 = -\{1 + \frac{2\phi\,(x^i)}{c^2}\}(c\,dt)^2 + \left\{1 - \frac{2\phi\,(x^i)}{c^2}\right\}(dx^2 + dy^2 + dz^2) \quad (24)$$

where the gravitational potential φ (x^i) is a function of position satisfying the Newtonian field equation and assumed to vanish at infinity.

Again, the fundamental matrices for the general metric are commonly diagonal. But this is not the case with Bondi metric used for the study of gravitational radiation which has non-zero off-diagonal elements. In this metric the co- ordinates used are (u, r, θ, φ) and the line element is [3]

$$ds^2 = \left(\frac{f}{r}e^{2\beta} - g^2r^2e^{2\alpha}\right)du^2 + 2e^{2\beta}du\,dr + 2gr^2e^{2\alpha}du\,d\theta - r^2(e^{2\alpha}d\theta^2 + e^{-2\alpha}Sin^2\theta d\phi^2) \quad (25)$$

Some uncommon line elements

Let us discuss here some uncommon line elements which are not widely used.

i) Alcubierre, in 1994, derived the following line element [2]

$$ds^2 = -dt^2 + [dx - V_s(t)f(r_s)dt]^2 + dy^2 + dz^2 \quad (26)$$

using the co-ordinates (t, x, y, z) and a curve $x= x_s(t)$, y=0, z=0 lying in the t-x plane which passes through the origin. Here, $V_s(t)= dx_s(t)/dt$ is the velocity associated with the curve and $r_s =[\{ x - x_s(t)\}^2 + y^2 + z^2]$. The function $f(r_s)$ is any smooth positive function that satisfies f(0) = 1 and continuously decreases away from the origin to vanish at $r_s> R$ for certain R.

ii) Let us consider the line element[2] of flat spae-time in the frame (t, x, y, z) rotating with an angular velocity ω about the z axis of an inertial frame

$$ds^2 = [1 - \omega^2(x^2 + y^2)]dt^2 + 2\omega(ydx - zdy) - dx^2 - dy^2 - dz^2 \quad (27)$$

Here, ds^2 is invariant [4].

iii) Sometimes an extra dimension over 4 is assumed in case of unified theory. The simplest case is a 5-D space-time in which the fifth dimension runs around a circle with a very small radius. The line element describing such a space-time is [2]

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + R^2d\psi^2 \quad (28)$$

where $0 \leq \psi \leq 2\pi$ and A, B, . . . range over 4. R is a constant which fixes the size of the circle.

Some well known line elements

Let us discuss some well known line elements which are also well used.

i) Schwarszchild line element – Here, we shall discuss the line element and solution of Schwarszchild in brief. K. Schwarszchild, in 1916, put forward exact exterior and interior solutions for spherically symmetric objects. Summarizing the Schwarszchild geometry the line element in gravitational unit is given by [2, 5]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \mathrm{Sin}^2\theta d\phi^2) \quad (29)$$

with $r > 2M$ or $r < 2M$ Schwarszchild geometry in Kruskal-Szekeres co-ordinates is given by [2]

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}}(-dv^2 + du^2) + r^2(d\theta^2 + \mathrm{Sin}^2\theta d\phi^2) \quad (30)$$

where r is a function of u and v.

It is to be mentioned that (29) has the following characteristics-

i) It is time independent, ii) It is spherically symmetric. So, the geometry of a 2-D surface of constant t and r could be obtained from it, iii) when $\frac{2M}{r} \to 0$, the co-efficient of dr^2 could be expanded and used in it without much loss of generality.

Now, to introduce Eddington- Finkelstein co-ordinates one might start from (29) for the case either with r>2M or r< 2M and substituting $t = v - r - 2M \log|\frac{r}{2M} - 1|$ one gets [2]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \mathrm{Sin}^2\theta\, d\phi^2 \quad (31)$$

We have to note that the form of the Schwarszchild line element depends on the system of co-ordinates used. For isotropic co-ordinates (r, θ, φ) we shall obtain [1, 3, 5]

$$ds^2 = -\left(1 + \frac{m}{2r_1}\right)^4 [{dr_1}^2 + r_1^2(d\theta^2 + \mathrm{Sin}^2\theta\, d\phi^2)] + \frac{(1 - \frac{m}{2r_1})^2}{(1 + \frac{m}{2r_1})^2} dt^2 \quad (32)$$

Again, the exterior solution for a spherically symmetric object could be given by the metric [1, 3 ,5]

$$ds^2 = -(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3})^{-1} dr^2 - r^2 d\theta^2 - r^2 \mathrm{Sin}^2\theta\, d\phi^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda r^2}{3}\right) c^2\, dt^2 \quad (33)$$

As the cosmological constant Λ is negligible, the term containing Λ may be neglected.

Also, for an object like a black hole Schwarszchild put forward a mathematical solution to the field equation in terms of the line element [3] which is similar to (33) without Λ.

ii) Kerr line element

Let us have the priviledge to discuss Kerr metric in brief. In 1963 R. P. Kerr derive the general relativistic exterior solution [8] for a spinning mass. Also, in 1965 Newman and Janis derived Kerr metric using null tetrad formation [9]. It is known that Kerr started from (33) and applying some conditions obtained the Kerr solution as[5]

$$ds^2 = \left[1 - \frac{2mr}{r^2 + a^2\mathrm{Cos}^2\theta}\right] c^2dt^2 - \left[\frac{r^2 + a^2\mathrm{Cos}^2\theta}{r^2 - 2mr + a^2}\right] dr^2 - [r^2 + a^2\mathrm{Cos}^2\theta]d\theta^2 - \left[(r^2 + a^2)\mathrm{Sin}^2\theta + \frac{2mra^2\mathrm{Sin}^4\theta}{r^2 + a^2\mathrm{Cos}^2\theta}\right] d\phi^2 - \left[\frac{4mra\mathrm{Sin}^2\theta}{r^2 + a^2\mathrm{Cos}^2\theta}\right] c\, dt\, d\phi \quad (34)$$

He, also, obtained the expression for Kerr metric having the following characteristics [3].

i) The geometry of space-time around a rotating massive body as well as the unusual new phenomena exhibited by rotating bodies are described by this equation. It may be mentioned that the rotation is due to curvature of space-time associated with the rotating body,

ii) It is an exact solution of Einstein field equations of general relativity and is nothing but a generalization of Schwarszchild metric,
iii) In standard spherical co-ordinates the metric, describing the space-time around a spinning black hole is relatively complicated and is merely stated as given by Boyer and Lindquist.

$$ds^2 = \left(1 - \frac{2mr}{\Sigma}\right) dt^2 + \frac{4mraSin^2\theta}{\Sigma} dt\, d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma\, d\theta^2 - \left(r^2 + a^2 + \frac{2a^2 mrSin^2\theta}{\Sigma}\right) Sin^2\theta\, d\phi^2 \quad ...(35)$$

iv) In the non-relativistic limit when m goes to zero (Schwarszchild radius r=2m also goes to zero) the Kerr metric becomes an orthogonal metric for the oblate spherical co-ordinates.
It should be mentioned that astronomical tests in a Kerr field have been derived and discussed in details by Krori and Barua [10].

iii) Line element for charged mass

It could be seen that the solution for a rotating mass (m) having charge (Q) was obtained by Newman and others [11] using null tetrad method. They started with the Reissneir-Nordström metric [5]

$$ds^2 = (1 - \frac{2mr - Q^2}{r^2} c^2 dt^2 - (1 - \frac{2mr - Q^2}{r^2})^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\, d\phi^2 \quad (36)$$

After introducing the rotation parameter ' a ' they constructed covariant and contravariant forms of the null tetrad satisfying some important properties. They assumed $\Delta = r^2 - 2mr + a^2 + Q^2$ and obtained the so called Kerr-Newman metric as

$$ds^2 = \left[1 - \frac{2mr - Q^2}{r^2 + a^2Cos^2\theta}\right] c^2\, dt^2 - \frac{r^2 + a^2Cos^2\theta}{\Delta} dr^2 - (r^2 + a^2Cos^2\theta) d\theta^2 - \left\{r^2 + a^2 + \frac{(2mr - Q^2)a^2Sin^2\theta}{r^2 + a^2Cos^2\theta}\right\} Sin^2\theta d\phi^2 - \frac{2a\,(2mr - Q^2)}{r^2 + a^2Cos^2\theta} Sin^2\theta\, c\, dt\, d\phi$$

(37)

which leads to R-N metric for a=0 and to Kerr metric for Q=0. Further Newman and others showed that (37) satisfies Einstein-Maxwell equations.

iv) R – N solution

Let us discuss about R-N solution in brief[3].
Hans Reissner in 1916 and Gunnar Nordström in 1918 derived independently an equation which is a modified version of Schwarszchild solution of Einstein's equation of general relativity valid for a radially symmetric body of mass m having acharge Q. This could be written as

$$ds^2 = f(r)c^2dt^2 - \frac{dr^2}{f(r)} - r^2(d\theta^2 + Sin^2\theta d\phi^2) \quad (38)$$

where $f(r) = 1 - \frac{2MG}{rc^2} + \frac{Q^2G}{r^2c^4}$.
Asanapplicationofrelativisticelectrodynamicsonemayfin dthegravitationalfieldof a static charged particle in general relativity where both gravitational and electric fields are spherically symmetric. In such a case, the line element (15) takes the form [5]

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{4\pi G\, Q^2}{c^4r^2}\right) c^2dt^2 - \left(1 - \frac{2m}{r} + \frac{4\pi G\, Q^2}{c^4r^2}\right)^{-1} dr^2 - r^2d\theta^2 - r^2Sin^2\theta\, d\phi^2 \quad (39)$$

It may be mentioned that Krori and Barua have derived a singularity-free interior solution for a charged fluid sphere [12].

Again, Lense and Thirring, in 1918, put forward an approximate exterior solution for a spinning sphere. It is [5]

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2dt^2 - \left(1 + \frac{2m}{r}\right)(dx^2 + dy^2 + dz^2) + \frac{4GJ}{c^3r} Sin^2\theta\, c\, dtd\phi \quad (40)$$

where $r^2 = x^2 + y^2 + z^2$ and J is the angular momentum of the sphere.
Now, for Penrose diagram [13] we want solution like that found by Kruskal when R-N metric is to be changed to [5]

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{4\pi G\, Q^2}{c^4r^2}\right) dv\, dw - r^2\,(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (41)$$

by substituting v = ct + r* and w = ct – r*.

v) Kruskal Solution

It has been shown that Kruskal solution [5] is maximal but is not geodesically complete [14] and is a transformation of the Schwarszchild solution. On substituting $v = c\,t + r + 2m\log_e(r-2m)$ and $w = c\,t - r - 2m\log_e(r-2m)$ where v and w are respectively known as advanced null and retarded null co-ordinates, we obtain $\mathrm{dv\,dw} = c^2 dt^2 - (1-\frac{2m}{r})^{-2} dr^2$ leading to the line element

$$ds^2 = \left(1-\frac{2m}{r}\right) dv\,dw - r^2\,(d\theta^2 + \ Sin^2\theta\ d\phi^2) \quad (42)$$

It is well known that the properties of homogeneity and isotropy apply only to the spatial part of the metric [3]. As the universe is expanding , the general form of the metric can be written as

$$ds^2 = dt^2 - a^2\,(t)\,d\sigma^2 \quad (43)$$

where $d\sigma^2$ is the spatial part of the metric and a(t) is the scale factor describing the expansion of this part of metric.

Now, Robertson-Walker metric [3]

$$ds^2 = dt^2 - \frac{a^2(t)}{1-kr^2} dr^2 - a^2(t) r^2 d\theta^2 - a^2(t) r^2 Sin^2\theta\ d\phi^2 \quad (44)$$

also describes a homogeneous, isotropic and expanding universe.

Again, on the basis of some assumptions R-W line element may be written as

$$ds^2 = dt^2 - \frac{{R_0}^2 e^{g(t)}}{\{1+\frac{kr^2}{4})^2} (dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta\ d\phi^2) \quad (45)$$

which represents the non-static cosmological model.

vi) Einstein's line element

It can be shown that Einstein's line element for static homogeneous universe takes the form [1, 3, 5]

$$ds^2 = dt^2 - (dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) \quad (46)$$

This form suggests that the spatial extent of the universe holds on the surface of a hyper-sphere in Euclidean space (z_1, z_2, z_3, z_4) of 4-D showing the isotropic and homogeneous nature of the universe. After some transformations it takes the form

$$ds^2 = -(1+\frac{\rho^2}{4R_0^2})^{-2}(dx^2 + dy^2 + dz^2) + dt^2 \quad (47)$$

vii) De Sitter Line Element

Over and above Einstein's line element we have another form of line element known as de Sitter line element which arises from the possibility that $p_0 + \rho_0 = 0$. It is for static, isotropic and homogeneous universe generally written as [1, 5]

$$ds^2 = -\,(1-\frac{r^2}{R_0^2})^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\ d\phi^2 + \left(1-\frac{r^2}{R_0^2}\right) dt^2 \quad (48)$$

It was modified by Lemaitre and Robertson independently and obtained [1]

$$ds^2 = -\,e^{-2kt}(dr^2 + r^2 d\theta^2 + r^2 Sin^2\theta\ d\phi^2) + dt^2 \quad (49)$$

We may mention that by using de Sitter model we can study the behavior of universe from the start. Matter and density of radiation will drop to negligible level as the universe is expanding. Also, this is a flat model for p=0 and ρ= 0. Hence [3],

$$ds^2 = dt^2 - a^2(t) dr^2 - a^2(t) r^2 d\theta^2 - a^2(t) r^2 Sin^2\theta\ d\phi^2 \quad (50)$$

Considering expansion of the universe qualitatively the line element may be written as

$$ds^2 = dt^2 - a^2(t) dr^2 - e^{2\sqrt{\frac{\Lambda}{3}}.r} r^2 d\theta^2 - a^2(t) r^2 Sin^2\theta\ d\phi^2 \quad (51)$$

Again, we have seen that the flat R-W metric [2] is

$$ds^2 = -\,dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \quad (52)$$

where a(t) is the scale factor and is a function of time also. This metric represents a homogeneous, isotropic cosmological model.

Considering isotropy and homogeneity of the universe Bianchi group of models are appropriate for solution. Bianchi type-1 metric may be taken in the form [5]

$$ds^2 = c^2dt^2 - X^2(t)\, dx^2 - Y^2(t)dy^2 - Z^2(t)dz^2 \quad (53)$$

viii) Robertson-Walker and Friedmann-Robertson-Walker Solution

F-R-W (or standard) cosmology was so called after the names of A. Friedmann, H. P. Robertson and A. G. Walker which is based on the metric called F-R-W metric. This metric is, again, based on Weyl postulates as well as cosmological principles.

It has been found that F-R-W metric is nothing but the result of some changes in Friedmann's metric [5] and reads as

$$ds^2 = c^2dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2 Sin^2\theta\, d\phi^2] \quad (54)$$

where k may be 0, ± 1.

Again, R-W line element is the non-static, spherically symmetric one in the moving co-ordinate system which, under some restrictions, is given by [1, 3, ss5]

$$ds^2 = dt^2 - \frac{{R_0}^2 e^{g(t)}}{1+\frac{kr^2}{4}}(dr^2 + r^2d\theta^2 + r^2 Sin^2\theta\, d\phi^2) \quad (55)$$

This relation would lead to Einstein's non-static line element for k= 1 which, again, leads to

$$ds^2 = dt^2 - e^{g(t)}(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2) \quad (56)$$

This means that the spatial geometry at any time t may be embedded in a 4-D Euclidean space (z_1 , z_2 , z_3 , z_4).

Again, for k=-1 and substituting $r = R_0 Sinh\, x$ (55) would give

$$ds^2 = dt^2 - R_0^2 e^{g(t)}[dx^2 + Sinh^2 x(d\theta^2 + Sin^2\theta\, d\phi^2) \quad (57)$$

ix) Solution of Vaidya and others

In 1951 the solution for a radiating optical object was derived by P. C. Vaidya. The line element representing Vaidya solution is [5]

$$ds^2 = \left\{1 - \frac{m(u)}{r}\right\} du^2 + 2du\, dr - r^2(d\theta^2 - Sin^2\theta\, d\phi^2) \quad (58)$$

Now, Vaidya, Krori and Barua found out, in 1974, the solution for a charged radiating sphere [15] following the method of Patino and Rago [16]. Again, Vaidya-Mallett solution [17] involves a rotating radiating charged mass embedded in a de Sitter universe neglecting Λ. Following metric [5] was obtained by them.

$$\begin{aligned} ds^2 = (1-\delta)du^2 &+ 2\, du\, dr - 2a\, Sin^2\theta\, dr\, d\phi \\ &- (\rho\rho^*)^{-1}d\theta^2 + 2\, a\, \delta\, Sin^2\theta\, du\, d\phi \\ &- Sin2\theta\, (r^2 + a^2 + a^2\delta\, Sin^2\theta)\, d\phi^2 \end{aligned} \quad (59)$$

Now, let us consider the metric [2]

$$ds^2 = -dt^2 + dr^2 + (b^2 + r^2)\, (d\theta^2 + Sin^2\theta\, d\phi^2 \quad (60)$$

where, b is a constant having dimensions of length. The characteristics of this metric are-

i) The space-time represented by it is not physically realistic. But, it is an easy way to introduce embedding diagram,

ii) It is similar to that of flat space-time in polar co-ordinates and shares a number of properties with it,

iii) It is independent of t,

iv) It is spherically symmetric because the surface obtained from it with constant r and t has the geometry of a sphere,

v) At very large r the space-time is approximately flat since the metric reduces to (12),

vi) Except for the value b=0 the geometry is curved in an interesting way.

x) Erez-Rosen Solution for an Oblate Spheroid

We know that stars and many other heavenly bodies will have quadruple moment due to their spinning motion. G. Erez and N. Rosen gave the solution of Einstein's field equations for an oblate spheroid [18].

They used Weyl-Levi-Civita line element with axial symmetry [5] and obtained

$$ds^2 = e^{2\Psi}c^2dt^2 - e^{2(\gamma-\Psi)}\left[\left(1+\frac{m^2\text{Sin}^2\theta}{r^2-2mr}\right)dr^2 + (r^2-2mr+m^2\text{Sin}^2\theta)d\theta^2\right] - e^{-2\Psi}(r^2 - 2mr)\text{Sin}^2\theta d\phi^2 \quad (61)$$

It is to be mentioned that geodesic study of the Erez-Rosen space-time has been made by Krori and Sarmah [19].

xi) Gravitational Collapse of a Dust Ball

It is known that a collapsing stellar body is similar to a collapsing dust ball. According to Birkhoff's theorem the exterior metric of the collapsing ball would be Schwarszchild exterior metric (33) neglecting ʌ. As the radius of the ball is a function of time so, we may write for the interior of the dust ball the flat F-R-W metric [5]

$$ds^2 = c^2dT^2 - R^2(T)[du^2 + u^2(d\theta^2 + \text{Sin}^2\theta\, d\phi^2)] \quad (62)$$

where T is the F-R-W time co-ordinate which is different from that of Schwarszchild. Here, u extends up to the radius of the dust ball and R(T) describes the physical motion of it [20].

It is to be mentioned that the line element representing the space-time around a rotating black hole having mass M and angular momentum J would be

$$ds^2 = -\left(1-\frac{2Mr}{\rho^2}\right)dt^2 - \frac{4Mar\text{Sin}^2\theta}{\rho^2}d\phi\, dt + \frac{\rho^2}{\Delta}dr^2 + \rho^2d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\text{Sin}^2\theta}{\rho^2}\right)\text{Sin}^2\theta d\phi^2 \quad (63)$$

where $a = \frac{J}{M}, \rho^2 = r^2 + a^2\text{Cos}^2\theta, \Delta = r^2 - 2Mr + a^2$. (t, r, θ, φ) used here are the Boyer-Lindquist co-ordinates which are analogous to Schwarszchild co-ordinates for a non-rotating black hole. The Kerr parameter 'a' has the dimension of length in geometrised units. The metric (63) is a solution of the vacuum Einstein equation.

Imposing some restrictions like θ = π/2 and ρ = r (63) becomes [2]

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 - \frac{4Ma}{r}d\phi\, dt + \frac{r^2}{\Delta}dr^2 + \left(r^2+a^2+\frac{2Ma^2}{r}\right)d\phi^2 \quad (64)$$

for the orbit of the equatorial plane.

xii) Red Shift of Light and Weyl's Hypothesis

To study the red shift of light we want a line element which has its basis on Weyl's hypothesis [3]. de Sitter line element (49) is to be used for this purpose. A shift of wavelength of the radiation emitted by a remote source, such as nebula or a radio galaxy [3], has been predicted by R-W line element (44). Now, putting ds = 0, dθ = dφ = 0, we get

$$\frac{dr}{dt} = \pm\frac{(1-kr^2)^{\frac{1}{2}}}{a(t)} \quad (65)$$

The ± sign holds respectively for light receding away and that proceeding towards the origin.

xiii) Linearised Gravitational Waves

Let us discuss about the linearised gravitational waves. Taking perturbations into account the line element for space-time is [2]

$$ds^2 = -dt^2 + [1+f(t-z)]dx^2 + [1-f(t-z)]dy^2 + dz^2 \quad (66)$$

where f(t-z) is a function under the condition |ρ(t-z)|≪ 1. This expression represents wave of curvature propagating in the positive z direction with velocity of light (c=1). It is to be noted that the size (amplitude) and shape of the wave propagating as ripples in curved space-time are determined by the function f.

To compute the effect produced due to the passage of gravitational wave in a region Maxwell equations are to be used. The gravitational waves may be looked upon as perturbations of Minkowski space time given by

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ where $|h_{\alpha\beta}| \ll 1$ so that the line element comes out to be [21]

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 + h_{\alpha\beta}dx^\alpha dx^\beta \quad (67)$$

where the perturbation corresponds to a wave travelling along the z axis. Hence, the line element becomes

$$ds^2 = c^2dt^2 - dz^2 - [1 + f_+(z - ct)]dx^2 - [1 + f_+(z - ct)]dy^2 + 2f_x(z - ct)dxdy \quad (68)$$

where (+) and (x) respectively refer to the two independent polarization characteristics of the gravitational waves in general relativity. (68) is a solution of Einstein's field equations having linear approximation in the so called Transverse Traceless Lorentz Gauge.

xiv) The metric g

Let us consider the metric g that introduces the scalar product in the tangent space and defines the line element so that with respect to the laboratory foliation co-frame it would read [22] (a, b, . . . =1, 2, 3,)

$$ds^2 = N\, d\sigma^2 + g_{ab}\underline{dx}^a\underline{dx}^b = N^2\, d\sigma^2 - {}^{(3)}g_{ab}\underline{dx}^a\underline{dx}^b \quad (69)$$

Here, $N^2 = g(n, n)$ is the length square of the foliation vector field n, $\underline{dx}^a = dx^a - n^a d\sigma$ is the transversal 3 co-vector basis. Then 3-metric${}^{(3)}g_{ab}$ is the positive definite Riemannian metric on the spatial 3-D slices corresponding to fixed values of the time σ. This metric defines the 3-D Hodge duality operator.

xv) Field of photon

Now, to find out the field of photon we may start with the metric [23]

$$ds^2 = -dr^2 - r^2d\theta^2 + 2du\, dv + 2Adu^2 \quad (70)$$

For axially symmetric plane-fronted gravitational waves where $0 \le r$, $0 \le \theta \le 2\pi$, $-\infty < u, v < +\infty$ and the co-ordinates (r, θ, v, u) are respectively 1, 2, 3, 4 and A is a function of r and u. The final result will be arrived at after taking some assumptions and few mathematical manipulations.

II. CONCLUSION

The study reveals that line elements applied to different cases are of different forms. Their types are also different. Considering the nature of the applications the line elements are derived accordingly. Further, this study would give an idea about a line element for a particular purpose which would lead to the solution of a specific problem. This work may be thought of as an archive of different types of line elements from which we may pick out one which would serve our purpose.

It is to note that further studies of the line elements are being continued and the results would be reported in time.

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