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# Some Properties of $P_g$ and Pairwise $P_g$ - Spaces

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## ABSTRACT
In this article, the separation axiom $R_0$ of Shanin and later rediscovered by Davis, has been generalized to $P_g$ - axiom using the concept of g-closure. The bitopological analogue of $P_g$ - axiom named as pairwise $P_g$ -axiom has been introduced and different characterizations and properties of $P_g$-space and pairwise $P_g$-space have been discussed.
**Keywords:** $P_g$ - axiom, g-closure, g-kernel, pairwise $P_g$ - axiom.

## I. INTRODUCTION

The separation axiom $R_0$ was introduced and studied by Shanin [8]. Later it was rediscovered by Davis in [1]. A space X is known as **$R_0$-space** if $x \notin cl\{y\}$ implies that $y \notin cl\{x\}$. We apply the notion of g-closure to define new separation axioms named as $P_g$-spaces by using g-closure in the definitions of $R_0$. In bitopological spaces pairwise-$R_0$ and pairwise-$R_1$ spaces were seen in [7] and pairwise-$R_0$ spaces have been studied in Misra and Dube [3].

In what follows, let $i, j \in \{1, 2\}$ and $i \neq j$.

## II. PRELIMINARIES

A subset A of X is said to be **g-closed** if $cl(A) \subset U$ whenever $A \subset U$ and U is open in (X, T) [2]. Clearly every closed set is g-closed. Complement of g-closed is called g-open. A set U is said to be **g-neighbourhood** of point $x \in X$ if $x \in U$ and U is g-open [4]. The family of all g-open (resp. g-closed) sets in a space (X, T) is denoted by GO (X, T) (resp. GC (X, T)). The g closure of a subset A in a space X, denoted by **gcl A** is defined as the intersection of all g-closed sets that contain A [2]. A space X is said to be **$g_1$** if for any two distinct points x and y of X there exists a g-open set U containing x but not y and a g-open set V containing y but not x [9]. A bitopological space $(X, T_1, T_2)$ is said to be **pairwise $g_1$** if for each pair of distinct points x, y of X, there is a $T_i$-g-open set U containing x but not y and a $T_j$-g-open V containing y but not x [9]. A bitopological space X is pairwise $R_0$ if for each $G \in T_i$, $x \in G$ implies $T_j$-cl $(\{x\}) \subset G$ [5]. A function $f : (X, T_1, T_2) \to (Y, T_1^*, T_2^*)$ is defined to be **pairwise continuous** if each of the functions between topological spaces $f : (X, T_1) \to (Y, T_1^*)$ and $f : (X, T_2) \to (Y, T_2^*)$ is continuous [6]. Similarly, pairwise closed is also defined.

**Lemma 2.1** [2]**:** If f: $X \to Y$ is a closed and continuous and if A is g-closed set in X, then f (A) is g-closed in Y.

**Lemma 2.2** [2]: If f: $X \to Y$ is a closed and continuous and if A is g-closed (res. g-open) set in Y, then $f^{-1}$ (A) is g-closed (res. g-open) in X.

## III. P g - SPACE

**Definition 3.1:** A space X is said to be a $P_g$-space if $x \notin cl\{y\}$ implies that $y \notin gcl\{x\}$. Clearly, every $R_0$ space is a $P_g$-space but converse is not true.

**Example 3.2:** Let X = {a, b, c}, T = {ϕ, {a}, {b, c}, X}, GC (X, T) = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, X}. Then the space X is a $P_g$-space but not a $R_0$-space.

**Theorem 3.3:** A space X is a $P_g$-space if and only if for each open set S and each $x \in S$, $gcl\{x\} \subseteq S$.

**Proof:** Let S be an open set containing x and let $y \notin S$. Then $x \notin cl\{y\}$. By $P_g$-axiom, $y \notin gcl\{x\}$. Thus $gcl\{x\} \subseteq S$.

**Conversely**, let $x \notin cl\{y\}$. Then there is an open set S (say) containing x, which has empty intersection with {y}, i.e. $y \notin S$. By hypothesis, $gcl\{x\} \subseteq S$ and thus, $y \notin gcl\{x\}$. Hence X is a $P_g$-space.

**Theorem 3.4:** For a space X, the following are equivalent:

(a) X is a $P_g$-space.

(b) For each $x \in X$, gcl{x} $\subseteq$ ker {x}.

(c) If F is a closed set in X, then F is the intersection of all the g-open sets containing F.

(d) If S is an open set in X, then S is the union of all the g-closed sets in X contained in S.

(e) For a non empty set A, and an open set S in X such that $S \cap A \neq \phi$, there is a g-closed set $F \subseteq S$ such that $F \cap A \neq \phi$.

(f) For any closed set F in X and $x \notin F$, gcl{x} $\cap F = \phi$.

**Proof:** (a) → (b): Let $y \in$ gcl{x} and S be an open set containing x. Since X is a $P_g$-space therefore by theorem 3.3, gcl{x} $\subseteq$ S and thus $y \in S$. Therefore $x \in$ gcl{y}, i.e. $y \in$ ker {x}. Hence gcl{x} $\subseteq$ ker{x}.

(b) → (c): Let F be a closed set. Let $x \notin F$. Then $X - F$ is an open set containing x. If $y \in$ gcl{x}, then from (b), $y \in$ ker{x} and therefore $x \in$ cl{y}. So $y \in X - F$. Hence, gcl{x} $\subseteq X - F$, which implies, $F \subseteq X -$ gcl{x} is a g-open set that does not contain x. Thus x does not belong to the intersection of all the g-open sets, which contain F. Hence (c) holds.

(c) → (d): By taking complements of (c), we get (d). (d) → (e): Since $S \cap A \neq \phi$, therefore let $x \in S \cap A$. Then $x \in$ open set S. Therefore, from (d), S is the union of all the g- closed sets of X contained in S. Hence there exists a closed set F (say) such that $x \in F \subseteq S$, which implies that $F \cap S \neq \phi$. Thus (e) holds.

(e) → (f): Let F be a closed set in X and $x \notin F$. Then $X - F$ is an open set in X such that $(X - F) \cap \{x\} \neq \phi$. Therefore, from (e), there is a g-closed set K such that $K \subseteq X - F$ and $K \cap \{x\} \neq \phi$. So gcl{x} $\subseteq X - F$. Hence gcl{x} $\cap F = \phi$. Thus (f) is true.

(f) → (a): Let S be an open set containing x. Then, from (f), we have $(X - S) \cap$ gcl{x} $= \phi$ and hence gcl{x} $\subseteq$ S. Thus by theorem 3.3, X is a $P_g$-space.

**Theorem 3.5:** A $P_g$-space X is $g_1$ if it is $T_0$.

**Proof:** Let $x \neq y \in T_0$-space X. Then, there exists an open set G containing x but not y. Since X is $P_g$-space therefore, by theorem 3.3, gcl{x} $\subseteq$ G. Therefore $y \notin$ gcl{x}. Take H = X – gcl{x} which is a g-open set containing y but not x. Also every open set is g-open. Thus g-open sets G and H satisfy the requirement of $g_1$-axiom for the space X.

**Theorem 3.6:** If f is a closed and continuous mapping from a $P_g$-space X to a Space Y, then Y is also a $P_g$-space.

**Proof:** Let $y_1$ and $y_2 \in Y$ and $y_1 \notin$ cl{$y_2$}. Then there exists an open set $V_1$ such that $y_1 \in V_1$ and $y_2 \notin V_1$. Put $f^{-1}(V_1) = G$. Since f is continuous therefore G is an open set in X. Also $f^{-1}(y_1) \in G$, $f^{-1}(y_2) \cap G = \phi$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Therefore $x_1 \notin$ cl{$x_2$}. By $P_g$-axiom on X, $x_2 \notin$ gcl{$x_1$}. Thus there is a g-open set $V_x$ in X containing $x_2$ but not $x_1$. $X -$ gcl {$x_1$} = $V_{x_2}$ (say) containing $x_2$ but not $x_1$.

Let $V = \cup \{V_{x_2} : x_2 \in f^{-1}(y_2)\}$. Then V is a g-open set in X containing $f^{-1}(y_2)$ but not $x_1$. So $X - V$ is a g-closed set in X. Since f is closed and continuous, therefore $f(X - V)$ is g-closed in Y containing $y_1$ but not $y_2$ [2]. Hence $Y - (f(X - V))$ is a g-open set in Y containing $y_2$ but not $y_1$. Hence, $y_2 \notin$ gcl {$y_1$}. Thus Y is a $P_g$-space.

## IV. PAIRWISE Pg- SPACE

**Definition 4.1:** A bitopological space $(X, T_1, T_2)$ is said to be pairwise $P_g$-space if $x \notin T_i$-cl{y} $\Rightarrow y \notin T_j$-gcl{x}.

**Example 4.2:** Let X = {a, b, c}, $T_1$ = {ϕ, {a}, {b}, {a, b}, X}, $T_2$ = {ϕ, {a, b}, X}.
GC (X, $T_1$) = {ϕ, {c}, {b, c}, {a, c}, X}, GC (X, $T_2$) = {ϕ, {c}, {b, c}, {a, c}, X}.
Then $(X, T_1, T_2)$ is a pairwise $P_g$-space.

**Theorem 4.3:** A space X is a pairwise $P_g$-space if and only if for each $T_i$-open set S and each $x \in S$, $T_j$-gcl{x} $\subseteq$ S.

**Proof:** Let S is a $T_i$-open set containing x and let $y \notin S$. Then $x \notin T_i$-cl{y}. Since X is a pairwise $P_g$-space, therefore $y \notin T_j$-gcl{x}. Hence $T_j$-gcl{x} $\subseteq$ S.

**Conversely,** Let $x \notin T_i$-cl{y}. So there is a $T_i$-open set S (say) containing x but not y. By hypothesis, $T_j$-gcl{x} $\subseteq$ S and thus $y \notin T_j$-gcl{x}. Hence X is a pairwise $P_g$-space.

**Theorem 4.4:** A space X is a pairwise $P_g$-space if and only if for each $x \in S$, $T_j$-gcl{x} $\subseteq T_i$-ker{x}.

**Proof:** Let $y \in T_j$-gcl{x}and S be a $T_i$-open set containing x. Since X is pairwise $P_g$, therefore, $T_j$-gcl{x} $\subseteq$ S. Hence $y \in S$. So $x \in T_i$-cl{y}, i.e. $y \in T_i$-ker{x}. Thus, $T_j$-cl{x} $\subseteq T_i$- ker{x}.

**Conversely,** Let $x \notin T_i$-cl{y}. Then $y \notin T_i$-ker{x}. Therefore, by hypothesis, $y \notin T_j$-gcl{x}. Hence X is a pairwise $P_g$-space.

**Theorem 4.5:** A pairwise $P_g$-space X is pairwise $g_1$ if it is pairwise $T_0$.

**Proof:** Let $x \neq y \in$ pairwise $T_0$-space. Then there exists a $T_i$-open set G containing x but not y. Since X is a pairwise $gR_0$-space by theorem 4.3 and the fact that every open set is g-open, $T_j$-gcl{x} $\subseteq$ G. Also $y \notin T_j$-gcl{x}. Take H = X − $T_j$-gcl{x}, which is a g-open set containing y but not x. Thus open sets G and H satisfy the requirement of pairwise $g_1$.

**Theorem 4.6:** If f: $(X, T_1, T_2) \rightarrow (Y, T_1^*, T_2^*)$ is a pairwise closed and pairwise continuous mapping from a $P_g$-space X to a Space Y, then Y is also a $P_g$-space.

**Proof:** Let $y_1$ and $y_2 \in Y$ and $y_1 \notin T_i^*$-cl{$y_2$}. Then there exists a $T_i^*$-open set $V_1$ such that $y_1 \in V_1$ and $y_2 \notin V_1$. Put $f^{-1}(V_1)$ = G. Since f is pairwise continuous therefore G is a $T_i$-open set in X. Also $f^{-1}(y_1) \in G$, $f^{-1}(y_2) \cap G = \phi$. Let $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Therefore, $x_1 \notin T_i$-cl{$x_2$}. By pairwise $P_g$-axiom on X, $x_2 \notin T_j$-gcl{$x_1$}. Thus, there is a $T_j$- g-open set $V_x$ in X containing $x_2$ but not $x_1$. X − $T_j$-gcl {$x_1$} = $V_{x_2}$ (say) containing $x_2$ but not $x_1$. Let V = $\cup\{V_{x_2} : x_2 \in f^{-1}(y_2)\}$. Then V is a $T_j$-g-open set in X containing $f^{-1}(y_2)$ but not $x_1$. So X − V is a $T_j$-g-closed set in X. Since f is pairwise closed and pairwise continuous, f(X − V) is $T_j^*$-g-closed in Y not containing $y_2$. Hence Y − (f(X − V)) is a $T_j^*$-g-open set in Y containing $y_2$ but not $y_1$. Hence $y_2 \notin T_j^*$-gcl {$y_1$}. Thus Y is a pairwise $P_g$-space.

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