© 2015 IJSRSET | Volume 1 | Issue 3 | Print ISSN : 2395-1990 | Online ISSN : 2394-4099
Themed Section: Engineering and Technology

# Application of HFS-FEM to Functionally Graded Materials

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## ABSTRACT

This paper presents an overview on applications of HFS-FEM to functionally graded materials. Recent developments on the hybrid fundamental solution (HFS) based finite element model (FEM) of steady-state heat transfer, transient heat conduction, nonlinear heat transfer, and elastic problems of functionally graded materials (FGMs) are described. Formulations for all cases are derived by means of modified variational functional and fundamental solutions. Generation of elemental stiffness equations from the modified variational principle is also discussed. Finally, a brief summary of the approach is provided.

**Keywords:** Finite Element Method, Fundamental Solution, Functionally graded material

## I. INTRODUCTION

FGMs are a class of relatively new and promising composite materials that have optimized material properties by combining different material components following a predetermined law [1-4]. They are heterogeneous composite materials with graded variation of constituents from one material phase to another, which results in continuously varying material properties. FGMs thus have superior thermal and mechanical performance to conventional homogeneous materials, and have a wide variety of engineering applications especially for the purpose of removing mismatches of thermo-mechanical properties between coating and substrate and reducing stress level in structures.

Recently, two effective numerical methods have beed developed for analysing mechanical performance of FGMs [5, 6]. The first is the so-called hybrid Trefftz FEM (or T-Trefftz method) [7-9]. Unlike in the conventional FEM, the T-Trefftz method couples the advantages of conventional FEM [10, 11] and BEM [12, 13]. In contrast to the standard FEM, the T-Trefftz method is based on a hybrid method which includes the use of an independent auxiliary inter-element frame field defined on each element boundary and an independent internal field chosen so as to a prior satisfy the homogeneous governing differential equations by means of a suitable truncated T-complete function set of homogeneous solutions. Since 1970s, T-Trefftz model has been considerably improved and has now become a highly efficient computational tool for the solution of complex boundary value problems. It has been applied to potential problems [14-17], two-dimensional elastics [18, 19], elastoplasticity [20, 21], fracture mechanics [22-24], micromechanics analysis [25], problem with holes [26, 27], heat conduction [6, 28-30], thin plate bending [31-34], thick or moderately thick plates [35-39], three-dimensional problems [40], piezoelectric materials [41-45], and contact problems [46-48].

On the other hand, the hybrid FEM based on the fundamental solution (F-Trefftz method for short) was initiated in 2008 [7, 49] and has now become a very popular and powerful computational methods in mechanical engineering. The F-Trefftz method is significantly different from the T-Trefftz method discussed above. In this method, a linear combination of the fundamental solution at different points is used to approximate the field variable within the element. The independent frame field defined along the element boundary and the newly developed variational functional are employed to guarantee the inter-element continuity, generate the final stiffness equation and establish linkage between the boundary frame field and internal field in the element. This review will focus on the F-Trefftz method.

The F-Trefftz method, newly developed recently [7, 49], has gradually become popular in the field of mechanical and physical engineering since it is initiated in 2008 [7, 50, 51]. It has been applied to potential problems [16, 52-54], plane elasticity [19, 55, 56], composites [57-60], piezoelectric materials [61-63], three-dimensional problems [64], functionally graded materials [5, 65-67], bioheat transfer problems [68-72], thermal elastic problems [73], hole problems [74, 75], heat conduction problems [49, 76], micromechanics problems [25, 77], and anisotropic elastic problems [78-80].

Following this introduction, the present review consists of six sections. T-Trefftz FEM steady-state heat transfer in FGMs is described in Section 2. It describes in detail the method of deriving element stiffness equations. Section 3 focuses on the essentials of F-Trefftz elements for transient heat conduction in FGMs. The applications of F-Trefftz elements to heat transfer in nonlinear FGMs and elastic anaysis are discussed in Sections 4-5. Finally, a brief summary of the developments of the Treffz methods is provided.

## II. Steady-state heat transfer in FGM

This section is concerned with the application of the T-Trefftz to the solution of Steady-state heat transfer in FGMs. A hybrid graded element model is described and used to analyse two-dimensional heat conduction problems in both isotropic and anisotropic exponentially graded materials.

### II.1 Basic formulations

Consider a 2D heat-conduction problem defined in an anisotropic inhomogeneous media:

$$\frac{\partial \tilde{K}_{ij}}{\partial X_i}\frac{\partial u(\mathbf{X})}{\partial X_j}+\tilde{K}_{ij}\frac{\partial^2 u(\mathbf{X})}{\partial X_i \partial X_j}=0 \qquad \forall \mathbf{X}\in\Omega \quad (1)$$

with the following boundary conditions:

—Specified temperature boundary condition

$$u=\bar{u} \qquad \text{on } \Gamma_u \quad (2)$$

—Specified heat flux boundary condition

$$q=-\tilde{K}_{ij}u_{,j}n_j=\bar{q} \qquad \text{on } \Gamma_q \quad (3)$$

where $\tilde{K}_{ij}$ denotes the thermal conductivity in terms of spatial variable $\mathbf{X}$ and is assumed to be symmetric and positive-definite ($\tilde{K}_{12}=\tilde{K}_{21}, \det\tilde{K}=\tilde{K}_{11}\tilde{K}_{22}-\tilde{K}_{12}^{2}>0$). $u$ is the sought field variable and $q$ represents the boundary heat flux. $n_j$ is the direction cosine of the unit outward normal vector $\mathbf{n}$ to the boundary $\Gamma=\Gamma_u\cup\Gamma_q$, and $\bar{u}$ and $\bar{q}$ are specified functions on the related boundaries, respectively. For convenience, the space derivatives are indicated by a comma, i.e. $u_{,j}=\partial u/\partial X_j$, and the subscript index $i, j$ takes values 1 and 2 in our analysis. Moreover, the repeated subscript indices stand for summation convention.

### II.2 Fundamental solution in FGMs

For simplicity, we assume the thermal conductivity varies exponentially with position vector, for example

$$\tilde{\mathbf{K}}(\mathbf{X})=\mathbf{K}\exp(2\boldsymbol{\beta}\cdot\mathbf{X}) \quad (4)$$

where vector $\boldsymbol{\beta}=(\beta_1,\beta_2)$ is a graded parameter and matrix $\mathbf{K}$ is symmetric and positive-definite with constant entries.

Substituting Eq (4) into Eq (81) yields

$$K_{ij}\partial_i\partial_j u(\mathbf{X})+2\beta_i K_{ij}\partial_j u(\mathbf{X})=0 \quad (5)$$

whose fundamental function defined in the infinite domain necessarily satisfies following equation

$$K_{ij}\partial_i\partial_j N(\mathbf{X},\mathbf{X}_s)+2\beta_i K_{ij}\partial_j N(\mathbf{X},\mathbf{X}_s)+\delta(\mathbf{X},\mathbf{X}_s)=0 \quad (6)$$

in which $\mathbf{X}$ and $\mathbf{X}_s$ denote arbitrary field point and source point in the infinite domain, respectively. $\delta$ is the Dirac delta function.

The closed-form solution to Eq (6) in two dimensions can be expressed as [81]

$$N(\mathbf{X},\mathbf{X}_s)=-\frac{K_0(\kappa R)}{2\pi\sqrt{\det\mathbf{K}}}\exp\{-\boldsymbol{\beta}\cdot(\mathbf{X}+\mathbf{X}_s)\} \quad (7)$$

where $\kappa=\sqrt{\boldsymbol{\beta}\cdot\mathbf{K}\boldsymbol{\beta}}$, $R$ is the geodesic distance defined as $R=R(\mathbf{X},\mathbf{X}_s)=\sqrt{\mathbf{r}\cdot\mathbf{K}^{-1}\mathbf{r}}$ and $\mathbf{r}=\mathbf{r}(\mathbf{X},\mathbf{X}_s)=\mathbf{X}-\mathbf{X}_s$. $K_0$ is the modified Bessel function of the second kind of zero order. For isotropic materials, $K_{12}=K_{21}=0$, $K_{11}=K_{22}=k_0>0$, (5) recasts as

$$k_0\nabla^2 u(\mathbf{X})+2k_0\beta_i\partial_i u(\mathbf{X})=0 \quad (8)$$

Then the fundamental solution given by (7) reduces to,

$$N(\mathbf{X},\mathbf{X}_s)=-\frac{K_0(\kappa R)}{2\pi k_0}\exp\{-\boldsymbol{\beta}\cdot(\mathbf{X}+\mathbf{X}_s)\} \quad (9)$$

which agrees with the result in [82].

### II.3 Generation of graded element

In this section, the procedure for developing a hybrid graded element model is described based on the boundary value problem (BVP) defined in Eqs (1)-(4). The focus is to fully introduce the smooth variation of material properties into element formulation, instead of stepwise constant approximation frequently used in the conventional FEM

Similar to T-Trefftz FEM, the main idea of the F-Trefftz approach is to establish an appropriate hybrid FE formulation whereby intra-element continuity is enforced on a nonconforming internal displacement field formed by a linear combination of fundamental solutions at points outside the element domain under consideration, while an auxiliary frame field is independently defined on the element boundary to enforce the field continuity across inter-element boundaries. But unlike in the HT FEM, the intra-element fields are constructed based on the fundamental solution defined in Eq (7), rather than T-functions. Consequently, a variational functional corresponding to the new trial function is required to derive the related stiffness matrix equation. With the problem domain divided into some sub-domains or elements denoted by $\Omega_e$ with the element boundary $\Gamma_e$, additional continuities are usually required on the common boundary $\Gamma_{Ief}$ between any two adjacent elements 'e' and 'f' (see Fig. 1):

$$\left.\begin{array}{ll} u_e = u_f & \text{(conformity)} \\ q_e + q_f = 0 & \text{(reciprocity)} \end{array}\right\} \quad \text{on } \Gamma_{Ief} = \Gamma_e \cap \Gamma_f \tag{10}$$

in the proposed hybrid FE approach.

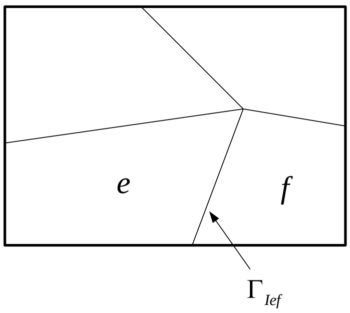


**Fig. 1** Illustration of continuity between two adjacent elements 'e' and 'f'

II.3.1 Non-conforming intra-element field

Activating by the idea of method of fundamental solution (MFS) [4] to remove the singularity of fundamental solution, for a particular element, say element $e$, which occupies sub-domain $\Omega_e$, we first assume that the field variable within an element is extracted from a linear combination of fundamental solutions centered at different source points (see Fig. 2), that is,

$$u_e(\mathbf{x}) = \sum_{j=1}^{n_s} N_e(\mathbf{x}, \mathbf{y}_j) c_{ej} = \mathbf{N}_e(\mathbf{x}) \mathbf{c}_e \tag{11}$$

where $c_{ej}$ is undetermined coefficients and $n_s$ is the number of virtual sources outside the element $e$. $N_e(\mathbf{x}, \mathbf{y}_j)$ is the required fundamental solution expressed in local element coordinates $(x_1, x_2)$, instead of global coordinates $(X_1, X_2)$ (see Fig. 2).

Evidently, Eq (11) analytically satisfies the heat conduction equation (5) due to the inherent property of $N_e(\mathbf{x}, \mathbf{y}_j)$.

In practice, the generation of virtual source points is usually done by means of the following formulation employed in the MFS [83-85]

$$\mathbf{y} = \mathbf{x}_b + \gamma(\mathbf{x}_b - \mathbf{x}_c) \tag{12}$$

where $\gamma$ is a dimensionless coefficient, $\mathbf{x}_b$ is the elementary boundary point and $\mathbf{x}_c$ is the geometrical centroid of the element. For a particular element shown in Fig. 2, we can use the nodes of element to generate related source points for simplicity.

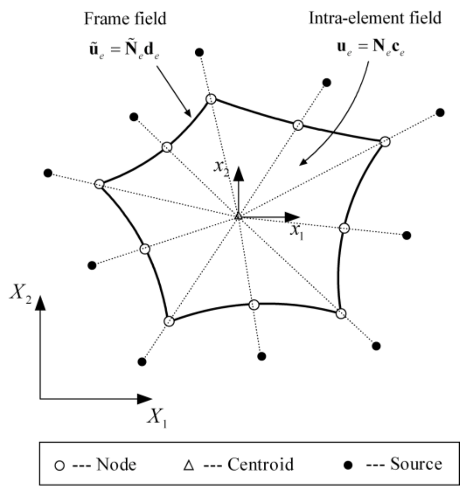


**Fig. 2** Intra-element field, frame field in a particular element in HFS-FEM, and generation of source points

The corresponding normal heat flux on $\Gamma_e$ is given by

$$q_e = -\tilde{\mathbf{K}}_{\mathbf{e}} \frac{\partial u_e}{\partial n} = \mathbf{Q}_e \mathbf{c}_e \quad (13)$$

where

$$\mathbf{Q}_e = -\tilde{\mathbf{K}}_{\mathbf{e}} \frac{\partial \mathbf{N}_e}{\partial n} = -\tilde{\mathbf{K}}_{\mathbf{e}} \mathbf{A} \mathbf{T}_e \quad (14)$$

with

$$\mathbf{T}_e = \left[\mathbf{N}_{e,1} \quad \mathbf{N}_{e,2}\right]^{\mathrm{T}} \qquad \mathbf{A} = \left[n_1 \quad n_2\right] \quad (15)$$

II.3.2 Auxiliary conforming frame field

In order to enforce the conformity on the field variable $u$, for instance, $u_e = u_f$ on $\Gamma_e \cap \Gamma_f$ of any two neighboring elements e and f, an auxiliary inter-element frame field $\tilde{u}$ is used and expressed in terms of the same degrees of freedom (DOF), $\mathbf{d}$, as used in the conventional finite elements. In this case, $\tilde{u}$ is confined to the whole element boundary

$$\tilde{u}_e(\mathbf{x}) = \tilde{\mathbf{N}}_e(\mathbf{x}) \mathbf{d}_e \quad (16)$$

which is independently assumed along the element boundary in terms of nodal DOF $\mathbf{d}_e$, where $\tilde{\mathbf{N}}_e$ represents the conventional FE interpolating functions. For example, a simple interpolation of the frame field on a side with three nodes of a particular element can be given in the form

$$\tilde{u} = \tilde{N}_1 u_1 + \tilde{N}_2 u_2 + \tilde{N}_3 u_3 \quad (17)$$

where $\tilde{N}_i$ ($i = 1, 2, 3$) stands for shape functions in terms of natural coordinate $\xi$ defined in Fig. 3.

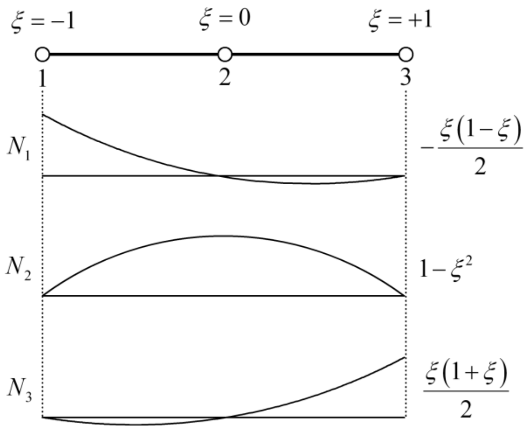


**Fig. 3** Typical quadratic interpolation for the frame field

II.3.2 Graded element

The fundamental solution for FGM is used as $N_e$ in Eq (11) to approximate the intra-element field. It can be seen from Eq (9) that $N_e$ varied throughout each element due to different geodesic distance for each field point, so the smooth variation of material properties can be achieved by this inherent property, instead of stepwise constant approximation frequently used in the conventional FEM, for example, Fig. 4 illustrates the two models when the thermal conductivity varies along direction $X_2$ in isotropic material.

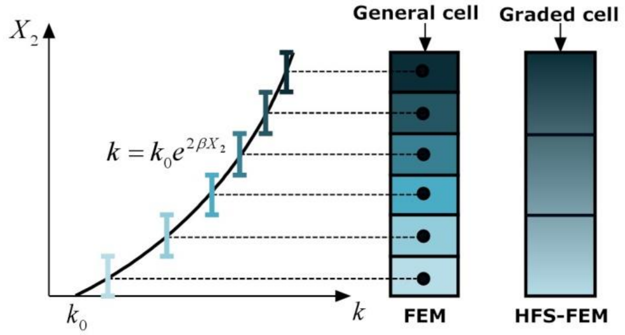


**Fig. 4** Comparison of computational cell in the conventional FEM and the proposed HFS-FEM

It should be mentioned here that Eq (4) which describes variation of the thermal conductivity is defined under global coordinate system. When contriving the intra-element field for each element, this formulation has to be transferred into local element coordinate defined at the center of the element, the graded matrix $\tilde{\mathbf{K}}$ in Eq (4) can, then, be expressed by

$$\tilde{\mathbf{K}}_e(\mathbf{x}) = \mathbf{K}_C \exp(2\boldsymbol{\beta} \cdot \mathbf{x}) \quad (18)$$

for a particular element e, where $\mathbf{K}_\mathrm{C}$ denotes the value of conductivity at the centroid of each element and can be calculated as follow:

$$\mathbf{K}_C = \mathbf{K} \exp(2\boldsymbol{\beta} \cdot \mathbf{X}_c) \quad (19)$$

where $\mathbf{X}_c$ is the global coordinate of the element centroid.

Accordingly, the matrix $\mathbf{K}_C$ is used to replace $\mathbf{K}$ (see Eq (7)) in the formulation of fundamental solution for FGM and the construction of intra-element field under local element coordinate for each element.

## II.4 Variational principle and stiffness equation

II.4.1 Modified functional

For the boundary value problem defined in Eqs (1)-(4), since the stationary conditions of the traditional potential or complementary variational functional can't guarantee the satisfaction of inter-element continuity condition required in the proposed HFS-FE model, a modified potential functional is developed as follows [49]

$$\Pi_m = \sum_e \left\{ -\int_{\Omega_e} \frac{1}{2} k u_{,i} u_{,i} \mathrm{d}\Omega - \int_{\Gamma_{qe}} \bar{q}\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} (\tilde{u} - u) q \mathrm{d}\Gamma \right\} \quad (20)$$

in which the governing equation (81) is assumed to be satisfied, a priori, in deriving the HFS-FE model. The boundary $\Gamma_e$ of a particular element consists of the following parts

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{qe} \cup \Gamma_{Ie} \quad (21)$$

where $\Gamma_{Ie}$ represents the inter-element boundary of the element 'e' shown in Fig. 1.

The stationary condition of the functional (20) can lead to the governing equation, boundary conditions and continuity conditions, which is shown here briefly. Eq (20) gives the following functional defined in a particular element:

$$\Pi_m = \sum_e [ -\frac{1}{2} \int_{\Omega_e} k u_{,i} u_{,i} \mathrm{d}\Omega - \int_{\Gamma_{qe}} \bar{q}\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} q(\tilde{u} - u) \mathrm{d}\Gamma ] \quad (22)$$

whose first-order variational yields

$$\begin{aligned} \delta\Pi_m = \sum_e [ & -\int_{\Omega_e} k u_{,i} \delta u_{,i} \mathrm{d}\Omega - \int_{\Gamma_{qe}} \bar{q}\delta\tilde{u} \mathrm{d}\Gamma \\ & + \int_{\Gamma_e} (\delta\tilde{u} - \delta u) q \mathrm{d}\Gamma + \int_{\Gamma_e} (\tilde{u} - u) \delta q \mathrm{d}\Gamma ] \end{aligned} \quad (23)$$

From the notation $q = -k u_{,i} n_i$ and the Gauss theorem

$$\int_\Omega h_{,i} \mathrm{d}\Omega = \int_\Gamma h n_i \mathrm{d}\Gamma \quad (24)$$

for any smooth function $h$ in the domain, we have

$$\begin{aligned} \delta\Pi_m = \sum_e [ & \int_{\Omega_e} (k u_{,i})_{,i} \delta u \mathrm{d}\Omega + \int_{\Gamma_{qe}} (q - \bar{q}) \delta\tilde{u} \mathrm{d}\Gamma \\ & + \int_{\Gamma_{ue}} q\delta\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_{Ie}} q\delta\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} (\tilde{u} - u) \delta q \mathrm{d}\Gamma ] \end{aligned} \quad (25)$$

For the displacement-based method, the potential conformity should be satisfied in advance, that is,

$$\begin{aligned} & \delta\tilde{u} = 0 && \text{on } \Gamma_{ue} && (\because \tilde{u} = \bar{u}) \\ & \delta\tilde{u}^e = \delta\tilde{u}^f && \text{on } \Gamma_{Ief} && (\because \tilde{u}^e = \tilde{u}^f) \end{aligned} \quad (26)$$

then, Eq (25) can be rewritten as

$$\begin{aligned} \delta\Pi_m = \sum_e [ & \int_{\Omega_e} (k u_{,i})_{,i} \delta u \mathrm{d}\Omega + \int_{\Gamma_{qe}} (q - \bar{q}) \delta\tilde{u} \mathrm{d}\Gamma \\ & + \int_{\Gamma_{Ie}} q\delta\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} (\tilde{u} - u) \delta q \mathrm{d}\Gamma ] \end{aligned} \quad (27)$$

from which the Euler equation in the domain $\Omega_e$ and boundary conditions on $\Gamma_e$ can be obtained

$$\begin{aligned} & (k u_{,i})_{,i} = 0 && \text{in } \Omega_e \\ & q = \bar{q} && \text{on } \Gamma_{qe} \\ & \tilde{u} = u && \text{on } \Gamma_e \end{aligned} \quad (28)$$

using the stationary condition $\delta\Pi_{me} = 0$.

II.4.2 Stiffness equation

Having independently defined the intra-element field and frame field in a particular element (see Fig. 2), the next step is to generate the element stiffness equation through a variational approach.

The variational functional $\Pi_e$ corresponding to a particular element $e$ of the present problem can be written as

$$\Pi_{me} = -\frac{1}{2} \int_{\Omega_e} k u_{,i} u_{,i} \mathrm{d}\Omega - \int_{\Gamma_{qe}} \bar{q}\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} q(\tilde{u} - u) \mathrm{d}\Gamma \quad (29)$$

Appling the Gauss theorem (24) again to the above functional, we have the following functional for the HFS-FE model

$$\begin{aligned} \Pi_{me} = & \frac{1}{2} \left[ \int_{\Gamma_e} q u \mathrm{d}\Gamma + \int_{\Omega_e} u (k u_{,i})_{,i} \mathrm{d}\Omega \right] \\ & - \int_{\Gamma_{qe}} \bar{q}\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} q(\tilde{u} - u) \mathrm{d}\Gamma \end{aligned} \quad (30)$$

Considering the governing equation (8), we finally have the functional defined on the element boundary only

$$\Pi_{me} = -\frac{1}{2} \int_{\Gamma_e} q u \mathrm{d}\Gamma - \int_{\Gamma_{qe}} \bar{q}\tilde{u} \mathrm{d}\Gamma + \int_{\Gamma_e} q\tilde{u} \mathrm{d}\Gamma \quad (31)$$

which yields by substituting Eqs (11), (13) and (16) into the functional (31)

$$\Pi_e = -\frac{1}{2} \mathbf{c}_e^{\mathrm{T}} \mathbf{H}_e \mathbf{c}_e - \mathbf{d}_e^{\mathrm{T}} \mathbf{g}_e + \mathbf{c}_e^{\mathrm{T}} \mathbf{G}_e \mathbf{d}_e \quad (32)$$

with

$$\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}} \mathbf{N}_e \mathrm{d}\Gamma, \ \mathbf{G}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}} \tilde{\mathbf{N}}_e \mathrm{d}\Gamma, \ \mathbf{g}_e = \int_{\Gamma_{qe}} \tilde{\mathbf{N}}_e^{\mathrm{T}} \bar{q} \mathrm{d}\Gamma \quad (33)$$

Next, to enforce inter-element continuity on the common element boundary, the unknown vector $\mathbf{c}_e$ should be expressed in terms of nodal DOF $\mathbf{d}_e$. The minimization of the functional $\Pi_e$ with respect to $\mathbf{c}_e$ and $\mathbf{d}_e$, respectively, yields

$$\frac{\partial \Pi_e}{\partial \mathbf{c}_e^{\mathrm{T}}} = -\mathbf{H}_e \mathbf{c}_e + \mathbf{G}_e \mathbf{d}_e = \mathbf{0}, \quad \frac{\partial \Pi_e}{\partial \mathbf{d}_e^{\mathrm{T}}} = \mathbf{G}_e^{\mathrm{T}} \mathbf{c}_e - \mathbf{g}_e = \mathbf{0} \quad (34)$$

from which the optional relationship between $\mathbf{c}_e$ and $\mathbf{d}_e$, and the stiffness equation can be produced

$$\mathbf{c}_e = \mathbf{H}_e^{-1} \mathbf{G}_e \mathbf{d}_e \quad \text{and} \quad \mathbf{K}_e \mathbf{d}_e = \mathbf{g}_e \quad (35)$$

where $\mathbf{K}_e = \mathbf{G}_e^{\mathrm{T}}\mathbf{H}_e^{-1}\mathbf{G}_e$ stands for the element stiffness matrix.

## III. Transient heat conduction in FGMs

### III.1 Statement of heat conduction problems in FGMs

Consider a two-dimensional (2D) transient heat conduction problem:

$$\nabla \cdot (k(X)\nabla u(X,t)) = \rho(X)c(X)\frac{\partial u(X,t)}{\partial t} \quad (36)$$

with the boundary conditions:

—Dirichlet boundary condition

$$u(X,t) = \bar{u}\ (X,t) \qquad \text{on } \Gamma_u \quad (37)$$

—Neumann boundary condition

$$q(X,t) = \bar{q}\ (X,t) \qquad \text{on } \Gamma_q \quad (38)$$

where $t$ denotes the time variable ( $t>0$ ). $k$ is the thermal conductivity dependent on the special variables $X \in \Omega \subset R^d$ . $d$ is the number of dimensions of the solution domain $\Omega$ ( $d=2$ in this study). $\rho$ is the mass density. $c$ is the specific heat, and $u$ is the unknown temperature field. $q$ represents the boundary heat flux defined by $q = -k\partial u/\partial n$ , where $n$ is the unit outward normal to the boundary $\Gamma = \Gamma_u \cup \Gamma_q$ . $\bar{u}$ and $\bar{q}$ are specified temperature and heat flow, respectively, on the related boundaries. In addition, an initial condition must be given for the time dependent problem. In this paper, a zero initial temperature distribution is considered, i.e.

$$u(X,0) = u_0(X) = 0 \quad (39)$$

The composition and the volume fraction of FGM constituents vary gradually with the coordinate X, giving a non-uniform microstructure with continuously graded macro-properties (conductivity, specific heat, density). In the present discussion, to make the derivation is tractable, the mass density is assumed to be constant within each element and taken the value of ρ at the centroid of the element. The thermal conductivity and specific heat have been chosen to have the same functional variation so that the thermal diffusivity $\eta$ is constant, that is

$$k(X) = k_0 f(X) \quad (40)$$

$$c(X) = c_0 f(X) \quad (41)$$

and

$$\eta = \frac{k_0}{c_0\rho} \quad (42)$$

It should be mentioned that the above assumption in FGMs leads to a class of solvable problems and can provide benchmark solutions to other numerical methods, such as FEM, meshless and BEM. Moreover, it can provide valuable insight into the thermal behavior of FGMs [86]. So this assumption has been followed by a lot of researchers in solving transient thermal problems in FGMs[4, 86].

### III.2 LT and fundamental solution in Laplace space

The LT of a function $u(X,t)$ is defined by

$$L(u(X,t)) = U(X,s) = \int_0^\infty u(X,t)e^{-st}dt \quad (43)$$

where $s$ is the Laplace parameter. By integration by parts, one can show that:

$$L[\frac{\partial u(X,t)}{\partial t}] = sU(X,s) - u_0(X) \quad (44)$$

The boundary conditions (37) and (38) become

$$U(X,s) = \frac{\bar{u}(X,t)}{s} \qquad \text{on } \Gamma_u \quad (45)$$

$$P(X,s) = \frac{\bar{q}}{s}\ (X,t) \qquad \text{on } \Gamma_q \quad (46)$$

III.2.1 Exponentially graded material

First, we consider a FGM with thermal conductivity and specific heat varying exponentially in one Cartesian coordinate, direction $X_2$ only,

$$k(X_2) = k_0 e^{2\beta X_2} \quad (47)$$

$$c(X_2) = c_0 e^{2\beta X_2} \quad (48)$$

where $\beta$ is the non-homogeneity graded parameter.

Substituting Eq. (47) and Eq. (48) into Eq. (36) yields

$$\nabla^2 u + 2\beta u_{X_2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \quad (49)$$

where $u_{X_2}$ denotes the derivative of $u$ with respect to $X_2$ ( $u_{X_2} = \partial u/\partial X_2$ )

After performing the LT, Eq. (49) becomes

$$\nabla^2 U + 2\beta U_{X_2} - \frac{s}{\alpha}U = 0 \quad (50)$$

in LT space ,where $u_0(X) = 0$ (at $t=0$ ) is considered (see Eq.(39)).

To obtain the fundamental solution of Eq. (50), following substitution is used here:

$$U = e^{-\beta X_2} G \tag{51}$$

In this case, the differential Eq.(50) in Laplace space becomes

$$\nabla^2 G - (\beta^2 + \frac{s}{\alpha})G = 0 \tag{52}$$

Obviously, Eq.(52) is the modified Helmholz equation, whose fundamental solution is

$$G = \frac{1}{2\pi} K_0(\sqrt{\beta^2 + \frac{s}{\alpha}} r) \tag{53}$$

Making use of Eq.(51), we obtain the fundamental solution of Eq. (50) in Laplace space

$$N(X, X_S) = \frac{1}{2\pi} e^{-\beta(X_2 - X_{S_2})} K_0(\sqrt{\beta^2 + \frac{s}{\alpha}} r) \tag{54}$$

where $r = \|X - X_S\|$, $X$ and $X_S$ denote arbitrary field point and source point in the infinite domain, respectively. $K_0$ is the modified Bessel function of the second kind of zero order.

III.2.2 General method for FGMs with different variation of properties

The method can be extended to a broader range of FGMs, not only exponential but also quadratic and trigonometric material variation, by variable transformations [86]. By defining a variable [86]

$$v(X,t) = \sqrt{k(X)} u(X,t) \tag{55}$$

Eq.(36) can be rewritten as

$$\nabla^2 v + (\frac{\nabla k(X) \cdot \nabla k(X)}{4k^2(X)} - \frac{\nabla^2 k(X)}{2k(X)}) v = \frac{\rho c(X)}{k(X)} \frac{\partial v}{\partial t} \tag{56}$$

For simplicity, define

$$k^{'}(X) = \frac{\nabla k(X) \cdot \nabla k(X)}{4k^2(X)} - \frac{\nabla^2 k(X)}{2k(X)} \tag{57}$$

Then, Eq. (56) can be rewritten as

$$\nabla^2 v + k^{'}(X) v = \frac{1}{\alpha} \frac{\partial v}{\partial t} \tag{58}$$

After performing the LT, the differential equation (58) becomes

$$\nabla^2 V + k^{'} V - \frac{s}{\alpha} V = 0 \tag{59}$$

When $k^{'}$ is a constant, Eq.(59) is a modified Helmholz equation whose fundamental solution is known. Then the fundamental solution of Eq.(36) in Laplace space can be obtained by inverse transformation:

$$N(X, X_S) = \frac{1}{2\pi} \frac{K_0(\sqrt{-k^{'} + \frac{s}{\alpha}} r)}{k(X)^{1/2} k(X_s)^{1/2}} \tag{60}$$

For quadratic material,

$$k(X) = k_0(a_1 + \beta X_2)^2 \tag{61}$$

In this case, $k^{'} = 0$ in Eq.(59).

For trigonometric material,

$$k(X) = k_0(a_1 \cos\beta X_2 + a_2 \sin\beta X_2)^2 \tag{62}$$

In this case, $k^{'} = \beta^2$ in Eq.(59).

For exponential material,

$$k(X) = k_0(a_1 e^{\beta X_2} + a_2 e^{-\beta X_2})^2 \tag{63}$$

In this case, $k^{'} = -\beta^2$ in Eq.(59). Substituting $k^{'} = -\beta^2$ into Eq.(60) and using the exponential law, the fundamental solution given by Eq.(60) reduces to Eq.(54).

Note that for quadratic, trigonometric and exponential variations of both heat conductivity and specific heat, the FGM transient problem can be transformed into the same differential equation which has a simple and standard form (Eq.(58)) [86].

### III.3 Generation of graded element

As in the conventional FEM, the solution domain is divided into sub-domains or elements. For a particular element, say element e, its domain is denoted by $\Omega_e$ and bounded by $\Gamma_e$. Since a nonconforming function is used for modeling the internal fields, additional continuities are usually required over the common boundary $\Gamma_{Ief}$ between any two adjacent elements 'e' and 'f' (see Fig. 1)[41]:

$$\left.\begin{array}{ll} U_e = U_f & \text{(conformity)} \\ P_e + P_f = 0 & \text{(reciprocity)} \end{array}\right\} \quad \text{on } \Gamma_{Ief} = \Gamma_e \cap \Gamma_f \tag{64}$$

in the proposed hybrid FE approach.

III.3.1 Non-conforming intra-element field

For a particular element, say element e, which occupies the sub-domain $\Omega_e$, the field variable within the element is extracted from a linear combination of fundamental solutions centered at different source points (see Fig. 2), that is,

$$U_e(x) = \sum_{j=1}^{n_s} N_e(x, x_S) c_{ej} = \mathbf{N}_e(x) \mathbf{c}_e \tag{65}$$

where $c_{ej}$ is undetermined coefficients and $n_s$ is the number of virtual sources outside the element e. $N_e\left(x,x_S\right)$ is the required fundamental solution expressed in local element coordinates $(x_1, x_2)$, rather than global coordinates $(X_1, X_2)$ (see Fig. 2). Clearly, Eq.(55) analytically satisfies the transformed governing equation of Eq.(36) in Laplace space due to the inherent property of $N_e\left(x,x_S\right)$.

The fundamental solution for FGMs ($N_e$ in Eq.(65)) is used to approximate the intra-element field for a FGM. The smooth variation of material properties throughout an element can be achieved by using the fundamental solution which can reflect the impact of a concentrated unit source acting at a point on any other points. The model inherits the essence of a FGM, so it can simulate FGMs more naturally than the stepwise constant approximation, which has been frequently used in conventional FEM. Fig. 3 illustrates the difference between the two models when the thermal conductivity varies along direction $X_2$.

Note that the thermal conductivity in Eq.(36) is defined in the global coordinate system. When contriving the intra-element field for each element, this formulation must be transferred into the local element coordinate system defined at the center of the element, and the graded heat conductivity $k(X)$ in Eq.(40) can then be expressed by

$$k_e(X) = k_C(X)\,f(X) \tag{66}$$

for a particular element e, where $k_C(X)$ denotes the value of conductivity at the centroid of each element and can be calculated as follows:

$$k_C(X) = k_0\,f(X_C) \tag{67}$$

where $X_C$ is the global coordinates of the element centroid.

Accordingly, $k_C$ is used to replace $k$ (see Eq.(60)) in the formulation of the fundamental solution for the FGM and to construct the intra-element field in the local element coordinate system for each element. In practice, the generation of virtual sources is usually achieved by means of the following formulation employed in the MFS [4]

$$y = x_b + \gamma\left(x_b - x_c\right) \tag{68}$$

where $\gamma$ is a dimensionless coefficient, $x_b$ and $x_c$ are, respectively, boundary point and geometrical centroid of the element. For a particular element as shown in Fig. 2, we can use the nodes of the element to generate related source points for simplicity.

The corresponding normal heat flux on $\Gamma_e$ is given by

$$P_e = -k_e \frac{\partial U_e}{\partial X_j} n_i = \mathbf{Q}_e \mathbf{c}_e \tag{69}$$

where

$$\mathbf{Q}_e = -k_e \frac{\partial \mathbf{N}_e}{\partial X_j} n_i = -k_e \mathbf{A}\mathbf{T}_e \tag{70}$$

with

$$\mathbf{T}_e = \begin{bmatrix}\mathbf{N}_{e,1} & \mathbf{N}_{e,2}\end{bmatrix}^{\mathrm{T}} \qquad \mathbf{A} = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \tag{71}$$

III.3.2 Auxiliary conforming frame field

In order to enforce conformity on the field variable $U$, for instance, $U_e = U_f$ on $\Gamma_e \cap \Gamma_f$ of any two neighboring elements e and f, an auxiliary inter-element frame field $\tilde{U}$ is used and expressed in terms of nodal degrees of freedom (DOF), $\mathbf{d}$, as used in conventional FEM as

$$\tilde{U}_e\left(x\right) = \tilde{\mathbf{N}}_e\left(x\right)\mathbf{d}_e \tag{72}$$

which is independently assumed along the element boundary, where $\tilde{\mathbf{N}}_e$ represents the conventional FE interpolating functions. For example, a simple interpolation of the frame field on the side with three nodes of a particular element can be given in the form

$$\tilde{U} = \tilde{N}_1\Psi_1 + \tilde{N}_2\Psi_2 + \tilde{N}_3\Psi_3 \tag{73}$$

where $\tilde{N}_i$ ($i = 1,2,3$) stands for shape functions which are the same as those in conventional FEM.

### III.4 Modified variational and stiffness equation

Having independently defined the intra-element field and frame field in a particular element (see Fig. 2), the element stiffness equation can be generated through a variational approach.

The final functional defined only on the element boundary is

$$\Pi_{me} = -\frac{1}{2}\int_{\Gamma_e} PU \mathrm{d}\Gamma - \int_{\Gamma_{qe}} \frac{\overline{q}}{s}\tilde{U}\mathrm{d}\Gamma + \int_{\Gamma_e} P\tilde{U}\mathrm{d}\Gamma \tag{74}$$

Substituting Eqs.(65), (69) and (72) into the functional (74), yields

$$\Pi_e = -\frac{1}{2}\mathbf{c}_e^{\mathrm{T}}\mathbf{H}_e\mathbf{c}_e - \mathbf{d}_e^{\mathrm{T}}\mathbf{g}_e + \mathbf{c}_e^{\mathrm{T}}\mathbf{G}_e\mathbf{d}_e \tag{75}$$

where

$$\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}} \mathbf{N}_e \mathrm{d}\Gamma, \ \mathbf{G}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}} \tilde{\mathbf{N}}_e \mathrm{d}\Gamma, \ \mathbf{g}_e = \int_{\Gamma_{qe}} \tilde{\mathbf{N}}_e^{\mathrm{T}} \frac{\overline{\mathrm{q}}}{s} \mathrm{d}\Gamma \quad (76)$$

Next, to enforce inter-element continuity on the common element boundary, the unknown vector $\mathbf{c}_e$ must be expressed in terms of nodal DOF $\mathbf{d}_e$. The minimization of the functional $\Pi_e$ with respect to $\mathbf{c}_e$ and $\mathbf{d}_e$, respectively, yields

$$\frac{\partial \Pi_e}{\partial \mathbf{c}_e^{\mathrm{T}}} = -\mathbf{H}_e \mathbf{c}_e + \mathbf{G}_e \mathbf{d}_e = \mathbf{0}, \quad \frac{\partial \Pi_e}{\partial \mathbf{d}_e^{\mathrm{T}}} = \mathbf{G}_e^{\mathrm{T}} \mathbf{c}_e - \mathbf{g}_e = \mathbf{0} \quad (77)$$

from which the optional relationship between $\mathbf{c}_e$ and $\mathbf{d}_e$, and the stiffness equation can be produced in the form

$$\mathbf{c}_e = \mathbf{H}_e^{-1} \mathbf{G}_e \mathbf{d}_e \quad \text{and} \quad \mathbf{K}_e \mathbf{d}_e = \mathbf{g}_e \quad (78)$$

where $\mathbf{K}_e = \mathbf{G}_e^{\mathrm{T}} \mathbf{H}_e^{-1} \mathbf{G}_e$ stands for the element stiffness matrix.

### III.5. Numerical inversion of LT

In this section, we present a brief review of the inversion of the LT used in this work. In general, once the solution for $U(X,s)$ in the Laplace space is found numerically by the method proposed above, inversion of the LT is needed to obtain the solution for $u(X,t)$ in the original physical domain. There are many inversion approaches for LT algorithms available in the literature [87]. A comprehensive review on those approaches can be found in [88]. In terms of numerical accuracy, computational efficiency and ease of implementation, Davies and Martin showed that Stehfest's algorithm gives good accuracy with a fairly wide range of functions [89]. Therefore, Stehfest's algorithm is chosen in our study.

If $F(s)$ is the LT of $f(t)$, an approximate value $f_a$ of the function $f(t)$ for a specific time $t = T$ is given by

$$f_a = \frac{\ln 2}{T} \sum_{i=1}^{N} V_i F\left(\frac{\ln 2}{T} i\right) \quad (79)$$

where

$$V_i = (-1)^{N/2+i} \sum_{k=\frac{i+1}{2}}^{\min(i,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(i-k)!(2k-i)!} \quad (80)$$

in which $N$ must be taken as an even number. In implementation, one should compare the results for different N to investigate whether the function is smooth enough, and determine an optimum N [87]. Stehfest suggested $N = 10$ and other researchers have found no significant change for $6 \le N \le 10$ [89]. Therefore, $N = 10$ is adopted here. That means that for each specific time $T$ it is necessary to solve different boundary value problems with different corresponding Laplace parameters $s = \frac{\ln 2}{T} i, i = 1, 2 \cdots 10$ in Laplace space 10 times, then weight and sum the solutions obtained in Laplace space.

## IV. F-Trefftz method for Nonlinear FGMs

### IV.1 Basic formulations

Consider a two-dimensional (2D) heat conduction problem defined in an anisotropic inhomogeneous media:

$$\sum_{i,j=1}^{2} \frac{\partial}{\partial X_i} \left( \tilde{K}_{ij}(\mathbf{X},\mathrm{u}) \frac{\partial u(\mathbf{X})}{\partial X_j} \right) = 0 \qquad \forall \mathbf{X} \in \Omega \quad (81)$$

For an inhomogeneous nonlinear functionally graded material, we assume the thermal conductivity varies exponentially with position vector and also be a function of temperature, that is

$$\tilde{K}_{ij}(\mathbf{X},u) = \alpha(u) K_{ij} \exp(2\boldsymbol{\beta} \cdot \mathbf{X}) \quad (82)$$

where $\alpha(u) > 0$ is a function of temperature which may be different for different materials, the vector $\boldsymbol{\beta} = (\beta_1, \beta_2)$ is a dimensionless graded parameter and matrix $\mathbf{K} = [K_{ij}]_{1 \le i,j \le 2}$ is a symmetric, positive-definite constant matrix ($K_{12} = K_{21}, \det K = K_{11}K_{22} - K_{12}{}^2 > 0$).

The boundary conditions are as follows:

—Dirichlet boundary condition

$$u = \bar{u} \qquad \text{on } \Gamma_u \quad (83)$$

—Neumann boundary condition

$$q = -\sum_{i,j=1}^{2} \tilde{K}_{ij} \frac{\partial u}{\partial X_j} n_i = \bar{q} \qquad \text{on } \Gamma_q \quad (84)$$

where $\tilde{K}_{ij}$ denotes the thermal conductivity which is the function of spatial variable $\mathbf{X}$ and unknown temperature field $u$. $q$ represents the boundary heat flux. $n_j$ is the direction cosine of the unit outward normal vector $\mathbf{n}$ to the boundary $\Gamma = \Gamma_u \cup \Gamma_q$. $\bar{u}$ and $\bar{q}$ are specified functions on the related boundaries, respectively.

### IV.2 Kirchhoff transformation and iterative method

Two methods are employed here to deal with the nonlinear term $\alpha(u)$, one is Kirchhoff transformation [90] and another is the iterative method.

(1) Kirchhoff transformation

$$\Psi(u)=\psi(u(\mathbf{X}))=\int\alpha(u)du \tag{85}$$

Making use of Eq.(85), Eq.(81) reduces to

$$\sum_{i,j=1}^{2}\frac{\partial}{\partial X_i}(K_{ij}^{*}(\mathbf{X})\frac{\partial\Psi(\mathbf{X})}{\partial X_j})=0 \quad \forall\mathbf{X}\in\Omega \tag{86}$$

where

$$K_{ij}^{*}(\mathbf{X})=K_{ij}\exp(2\boldsymbol{\beta}\cdot\mathbf{X}) \tag{87}$$

Substituting Eq.(87) into Eq.(86) yields

$$\left[\sum_{i,j=1}^{2}K_{ij}\frac{\partial^2\Psi(\mathbf{X})}{\partial X_i\partial X_j}+2\boldsymbol{\beta}\cdot(\mathbf{K}\nabla\Psi(\mathbf{X}))\right]\exp(2\boldsymbol{\beta}\cdot\mathbf{X})=0 \tag{88}$$

where

$$u=\psi^{-1}(\Psi) \tag{89}$$

It should be mentioned that the inverse of $\Psi$ in Eq.(89) exists since $\alpha(u)>0$.

The fundamental solution to Eq.(88) in two dimensions can be expressed as [90]

$$N(\mathbf{X},\mathbf{X}_s)=-\frac{K_0(\kappa R)}{2\pi\sqrt{\det\mathbf{K}}}\exp\{-\boldsymbol{\beta}\cdot(\mathbf{X}+\mathbf{X}_s)\} \tag{90}$$

where $\kappa=\sqrt{\boldsymbol{\beta}\cdot\mathbf{K}\boldsymbol{\beta}}$, $R$ is the geodesic distance defined as $R=R(\mathbf{X},\mathbf{X}_s)=\sqrt{\mathbf{r}\cdot\mathbf{K}^{-1}\mathbf{r}}$ and $\mathbf{r}=\mathbf{X}-\mathbf{X}_s$ in which $\mathbf{X}$ and $\mathbf{X}_s$ denote observing field point and source point in the infinite domain, respectively. $K_0$ is the modified Bessel function of the second kind of zero order. For isotropic materials, $K_{12}=K_{21}=0$, $K_{11}=K_{22}=k_0>0$, then the fundamental solution given by (90) reduces to

$$N(\mathbf{X},\mathbf{X}_s)=-\frac{K_0(\kappa R)}{2\pi k_0}\exp\{-\boldsymbol{\beta}\cdot(\mathbf{X}+\mathbf{X}_s)\} \tag{91}$$

which agrees with the result in [82].

Under the Kirchhoff transformation, the boundary conditions (83)-(84) are transformed into the corresponding boundary conditions in terms of $\Psi$.

$$\Psi=\psi(\bar{u}) \quad \text{on } \Gamma_u \tag{92}$$

$$p=-\sum_{i,j=1}^{2}K_{ij}^{*}\frac{\partial\Psi}{\partial X_j}n_i=-\sum_{i,j=1}^{2}\tilde{K}_{ij}\frac{\partial u}{\partial X_j}n_i=q=\bar{q} \text{ on } \Gamma_q \tag{93}$$

Therefore, by Kirchhoff transformation, the original nonlinear heat conduction equation (81), in which the heat conductivity is a function of coordinate $X$ and unknown function $u$, can be transformed into the linear equation (86) in which the heat conductivity is just a function of coordinate $X$. At the same time, the field variable becomes $\Psi$ in Eq.(86), rather than $u$ in Eq.(81). The boundary conditions (83)-(84) are correspondingly transformed into Eqs.(92)-(93). Once $\Psi$ is determined, the temperature solution $u$ can be found by the reversion of transformation (89), i.e. $u=\psi^{-1}(\Psi)$.

(2) Iterative method

Since the heat conductivity depends on the unknown function $u$, an iterative procedure is employed for determining the temperature distribution. The algorithm is given as follows:

1. Assume an initial temperature $u^0$.
2. Calculate the heat conductivity in Eq.(82) using $u^0$.
3. Solve the boundary value problem defined by Eqs.(81)-(84) for the temperature u
4. Define the convergent criterion $|u-u^0|<\delta$ ($\delta=10^{-6}$ in our analysis). If the criterion is satisfied, output the result and terminate the process. If not satisfied, go to next step.
5. Update $u^0$ with $u$
6. Go to step 2.

### IV.3 Generation of graded element

In this section, an element formulation is presented to deal with materials with continuous variation of physical properties. Such an element model is usually known as a hybrid graded element which can be used for solving the boundary value problem (BVP) defined in Eqs.(86) and (92)-(93). As was done in conventional FEM, the solution domain is divided into sub-domains or elements. For a particular element, say element e, its domain is denoted by $\Omega_e$ and bounded by $\Gamma_e$. Since a nonconforming function is used for modeling intra-element field, additional continuities are usually required over the common boundary $\Gamma_{Ief}$ between any two adjacent elements 'e' and 'f' (see Fig. 1):

$$\left.\begin{array}{ll}\Psi_e=\Psi_f & \text{(conformity)}\\ p_e+p_f=0 & \text{(reciprocity)}\end{array}\right\} \text{ on } \Gamma_{Ief}=\Gamma_e\cap\Gamma_f \tag{94}$$

in the proposed hybrid FE approach.

IV.3.1 Non-conforming intra-element field

For a particular element, say element e, which occupies sub-domain $\Omega_e$, the field variable within the element is extracted from a linear combination of fundamental solutions centered at different source points (see Fig. 2) that

$$\Psi_e(\mathbf{x}) = \sum_{j=1}^{n_s} N_e(\mathbf{x}, \mathbf{y}_j) c_{ej} = \mathbf{N}_e(\mathbf{x})\mathbf{c}_e \quad \forall \mathbf{x} \in \Omega_e, \mathbf{y}_j \notin \Omega_e \tag{95}$$

where $c_{ej}$ is undetermined coefficients and $n_s$ is the number of virtual sources outside the element e. $N_e(\mathbf{x}, \mathbf{y}_j)$ is the required fundamental solution expressed in terms of local element coordinates $(x_1, x_2)$, instead of global coordinates $(X_1, X_2)$ (see Fig. 2). Obviously, Eq. (95) analytically satisfies the heat conduction equation (88) due to the inherent property of $N_e(\mathbf{x}, \mathbf{y}_j)$.

The fundamental solution for FGM ($N_e$ in Eq.(95)) is used to approximate the intra-element field in FGM. It is well known that the fundamental solution represents the filed generated by a concentrated unit source acting at a point, so the smooth variation of material properties throughout an element can be achieved by this inherent property, instead of the stepwise constant approximation, which has been frequently used in the conventional FEM. For example, Fig. 4 illustrates the difference between the two models when the thermal conductivity varies along direction $X_2$ in isotropic material.

Note that the thermal conductivity in Eq. (87) is defined in the global coordinate system. When contriving the intra-element field for each element, this formulation has to be transferred into local element coordinate system defined at the center of the element, the graded matrix $\mathbf{K}^*$ in Eq. (87) can, then, be expressed by

$$\mathbf{K}_e^*(\mathbf{x}) = \mathbf{K}_C \exp(2\boldsymbol{\beta} \cdot \mathbf{x}) \tag{96}$$

for a particular element e, where $\mathbf{K}_C$ denotes the value of conductivity at the centroid of each element and can be calculated as follows:

$$\mathbf{K}_C = \mathbf{K} \exp(2\boldsymbol{\beta} \cdot \mathbf{X}_c) \tag{97}$$

where $\mathbf{X}_c$ is the global coordinates of the element centroid.

Accordingly, the matrix $\mathbf{K}_C$ is used to replace $\mathbf{K}$ (see Eq.(90)) in the formulation of fundamental solution for FGM and to construct intra-element field in the coordinate system local to element.

In practice, the generation of virtual sources is usually done by means of the following formulation employed in the MFS [4]

$$\mathbf{y} = \mathbf{x}_b + \mu(\mathbf{x}_b - \mathbf{x}_c) \tag{98}$$

where $\mu$ is a dimensionless coefficient ($\mu$=2.5 in our analysis[4]), $\mathbf{x}_b$ and $\mathbf{x}_c$ are, respectively, boundary point and geometrical centroid of the element. For a particular element shown in Fig. 2, we can use the nodes of element to generate related source points.

The corresponding normal heat flux on $\Gamma_e$ is given by

$$p_e = -\mathbf{K}_e^* \frac{\partial \Psi_e}{\partial X_j} n_i = \mathbf{Q}_e \mathbf{c}_e \tag{99}$$

where

$$\mathbf{Q}_e = -\mathbf{K}_e^* \frac{\partial \mathbf{N}_e}{\partial X_j} n_i = -\mathbf{A}\mathbf{K}_e^* \mathbf{T}_e \tag{100}$$

with

$$\mathbf{T}_e = \begin{bmatrix} \mathbf{N}_{e,1} & \mathbf{N}_{e,2} \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{A} = \begin{bmatrix} n_1 & n_2 \end{bmatrix} \tag{101}$$

IV.3.2 Auxiliary conforming frame field

In order to enforce the conformity on the field variable $u$, for instance, $\Psi_e = \Psi_f$ on $\Gamma_e \cap \Gamma_f$ of any two neighboring elements e and f, an auxiliary inter-element frame field $\tilde{\Psi}$ is used and expressed in terms of nodal degrees of freedom (DOF), $\mathbf{d}$, as used in the conventional finite elements as

$$\tilde{\Psi}_e(\mathbf{x}) = \tilde{\mathbf{N}}_e(\mathbf{x})\mathbf{d}_e \tag{102}$$

which is independently assumed along the element boundary, where $\tilde{\mathbf{N}}_e$ represents the conventional FE interpolating functions. For example, a simple interpolation of the frame field on the side with three nodes of a particular element can be given in the form

$$\tilde{\Psi} = \tilde{N}_1 \Psi_1 + \tilde{N}_2 \Psi_2 + \tilde{N}_3 \Psi_3 \tag{103}$$

where $\tilde{N}_i$ ($i = 1, 2, 3$) stands for shape functions in terms of natural coordinate $\xi$ defined in Fig. 3.

**IV.4 Modified variational principle and stiffness equation**

IV.4.1 Modified variational functional

For the boundary value problem defined in Eqs.(86) and (92)-(93), since the stationary conditions of the traditional potential or complementary variational functional can't guarantee the satisfaction of inter-element continuity condition required in the proposed HFS-FE model, a modified potential functional is developed as follows [7]

$$\begin{aligned}\Pi_m = \sum_e \Pi_{me} = \sum_e [ -\int_{\Omega_e} \frac{1}{2} K_{ij}^* \Psi_{,i} \Psi_{,j} \mathrm{d}\Omega \\ -\int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} \left(\tilde{\Psi} - \Psi\right) p \mathrm{d}\Gamma ]\end{aligned} \tag{104}$$

in which the governing equation (86) is assumed to be satisfied, a priori, in deriving the HFS-FE model (For convenience, the repeated subscript indices stand for summation convention). The boundary $\Gamma_e$ of a particular element consists of the following parts

$$\Gamma_e = \Gamma_{ue} \cup \Gamma_{qe} \cup \Gamma_{Ie} \tag{105}$$

where $\Gamma_{Ie}$ represents the inter-element boundary of the element 'e' shown in Fig. 1.

The stationary condition of the functional (104) can lead to the governing equation (Euler equation), boundary conditions and continuity conditions, details of the derivation can refer to Ref. [7].

IV.4.2 Stiffness equation

Having independently defined the intra-element field and frame field in a particular element (see Fig. 2), the next step is to generate the element stiffness equation through a variational approach and to establish a linkage between the two independent fields.

The variational functional $\Pi_e$ corresponding to a particular element e of the present problem can be written as

$$\begin{aligned}\Pi_{me} = -\frac{1}{2}\int_{\Omega_e} K_{ij}^* \Psi_{,i} \Psi_{,j} \mathrm{d}\Omega \\ -\int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\left(\tilde{\Psi} - \Psi\right)\mathrm{d}\Gamma\end{aligned} \tag{106}$$

Appling the Gauss theorem to the above functional, we have the following functional for the HFS-FE model

$$\begin{aligned}\Pi_{me} = \frac{1}{2}\left[\int_{\Gamma_e} p\Psi\mathrm{d}\Gamma + \int_{\Omega_e} \Psi (K_{ij}^* u_{,i})_{,j} \mathrm{d}\Omega\right] \\ -\int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\left(\tilde{\Psi} - \Psi\right)\mathrm{d}\Gamma\end{aligned} \tag{107}$$

Considering the governing equation (86), we finally have the functional defined on the element boundary only

$$\Pi_{me} = -\frac{1}{2}\int_{\Gamma_e} p\Psi\mathrm{d}\Gamma - \int_{\Gamma_{qe}} \bar{q}\tilde{\Psi}\mathrm{d}\Gamma + \int_{\Gamma_e} p\tilde{\Psi}\mathrm{d}\Gamma \tag{108}$$

which yields by substituting Eqs (95), (99) and (102) into the functional (108)

$$\Pi_e = -\frac{1}{2}\mathbf{c}_e^{\mathrm{T}}\mathbf{H}_e\mathbf{c}_e - \mathbf{d}_e^{\mathrm{T}}\mathbf{g}_e + \mathbf{c}_e^{\mathrm{T}}\mathbf{G}_e\mathbf{d}_e \tag{109}$$

with

$$\mathbf{H}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}}\mathbf{N}_e\mathrm{d}\Gamma,\ \mathbf{G}_e = \int_{\Gamma_e} \mathbf{Q}_e^{\mathrm{T}}\tilde{\mathbf{N}}_e\mathrm{d}\Gamma,\ \mathbf{g}_e = \int_{\Gamma_{qe}} \tilde{\mathbf{N}}_e^{\mathrm{T}}\bar{\mathrm{q}}\mathrm{d}\Gamma \tag{110}$$

## V. Elastic problems in FGMs

### V.1 Formulation of the problem

In this section, basic equations and the corresponding fundamental solutions for FGMs presented in [91] are briefly reviewed to provide notations and references for the subsequent sections.

V.1.1 Basic equations

For a 2D linear elastic problem, the governing equations of force equilibrium in the absence of body forces are given by

$$\sigma_{ij,j} = 0 \tag{111}$$

where $\sigma_{ij}$ are the components of the Cauchy stress tensor. For plane problems, all indices range from 1 to 2 and an index followed by a comma stands for partial differentiation with respect to the spatial coordinate. The summation convention is implied for repeated indices.

For the functionally graded materials considered in this study, the elastic stiffness tensor $C_{ijkl}$ is associated with the spatial variable $\mathbf{x} = (x_1, x_2)$ ; that is, $C_{ijkl} = C_{ijkl}(\mathbf{x})$ . Therefore, the linear elastic strain-stress relation is written as

$$\sigma_{ij} = C_{ijkl}(\mathbf{x})\varepsilon_{kl} \tag{112}$$

The components of stiffness tensor $C_{ijkl}$ must satisfy the usual symmetric condition

$$C_{ijkl} = C_{klij} = C_{ijlk} = C_{jikl} \tag{113}$$

Specially, for isotropic inhomogeneous elastic media, the elastic stiffness tensor $C_{ijkl}$ is written as

$$C_{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ij}\delta_{kl} + \mu(\mathbf{x})\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{114}$$

where $\delta_{ij}$ is Kronecker's delta, the Lame elastic parameters $\lambda(\mathbf{x})$ and $\mu(\mathbf{x})$ are the functions of spatial coordinate variable $\mathbf{x}$ and can be expressed in terms of elastic modulus $E$ , and Poisson ratio $\nu$ as

$$\lambda(\mathbf{x}) = \frac{3-\kappa}{\kappa-1}\mu(\mathbf{x}), \qquad \mu(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1+\nu)} \quad (115)$$

with $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

Therefore, the constitutive law Eq. (112) can be simplified as

$$\sigma_{ij} = \lambda(\mathbf{x})\delta_{ij}\varepsilon_{kk} + 2\mu(\mathbf{x})\varepsilon_{ij} \quad (116)$$

As well, the infinitesimal strain tensor $\varepsilon_{ij}$ related to the displacement field is expressed as

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad (117)$$

Substituting Eq. (117) into the constitutive equation (116) and then into the equilibrium equation (111) we have

$$\begin{aligned}&\lambda_{,i}(\mathbf{x})u_{j,j}(\mathbf{x}) + \{\lambda(\mathbf{x}) + \mu(\mathbf{x})\}u_{j,ji}(\mathbf{x})\\ &+\mu(\mathbf{x})u_{i,jj}(\mathbf{x}) + \mu_{,j}(\mathbf{x})\{u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x})\} = 0\end{aligned} \quad (118)$$

If the material is homogeneous, i.e., the Lame parameters are independent of the spatial variable $\mathbf{x}$ , Eq. (118) becomes

$$\{\lambda + \mu\}u_{j,ji}(\mathbf{x}) + \mu u_{i,jj}(\mathbf{x}) = 0 \quad (119)$$

which is the classic Navier-Cauchy equation with respect to displacements.

The boundary conditions have the same form as those of homogeneous materials:

$$\begin{aligned} u_i &= \bar{u}_i \quad \text{on } \Gamma_u \\ t_i &= \bar{t}_i \quad \text{on } \Gamma_t \end{aligned} \quad (120)$$

where $t_i = \sigma_{ij}n_j$ represents the ith component of the boundary traction, and $n_i$ is the ith component of outward normal to the boundary. $\Gamma_u$ and $\Gamma_t$ are the boundaries on which the displacement and the traction are prescribed respectively. An overbar denotes that the variable is specified.

V.1.2 Fundamental solutions for quadratic variation of elasticity

In this work, the Lame constants $\lambda$ and $\mu$ are assumed to be quadratic variation of the spatial variable $\mathbf{x}$ , that is

$$\lambda(\mathbf{x}) = \lambda_0\left(c + \beta_i x_i\right)^2, \qquad \mu(\mathbf{x}) = \mu_0\left(c + \beta_i x_i\right)^2 \quad (121)$$

where $c$ , $\lambda_0$ and $\mu_0$ are the corresponding material constants, $\beta_i$ is a graded parameter, which has a dimension of $\mathrm{m}^{-1}$ . In particular, if the graded parameter $\beta_i$ is equal to zero, the Lame constants in Eq. (121) will be reduced to two constants, and then the system of partial differential equations (118) will be the standard Navier-Cauchy equations for homogeneous isotropic elastic materials.

According to the work of [92], when the Poisson ratio $\nu$ is equal to 0.25 (a rather common value for rock materials) and the plane strain state is considered, one obtains

$$\lambda_0 = \mu_0 \quad (122)$$

which can significantly simplify the derivation of fundamental solutions.

Generally, the free space fundamental displacement solution for an isotropic inhomogeneous elastic continuum must satisfy the following equation system

$$\begin{aligned}&\lambda_{,i}(\mathbf{x})u_{j,j}(\mathbf{x}) + \{\lambda(\mathbf{x}) + \mu(\mathbf{x})\}u_{j,ji}(\mathbf{x}) + \mu(\mathbf{x})u_{i,jj}(\mathbf{x})\\ &+\mu_{,j}(\mathbf{x})\{u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x})\} + \delta\left(\mathbf{x}-\mathbf{x}^s\right)e_i = 0\end{aligned} \quad (123)$$

where $\mathbf{x}$ is a field point in the infinite plane, $\mathbf{x}^s$ is a source point at which the unit force $e_i$ along the i-direction is applied, and $\delta(\mathbf{x})$ is the Dirac delta function.

To obtain the fundamental displacement solution for the equilibrium equations (123) the following transformation is established for the displacement vector [91]

$$w_i = \left(c + \beta_k x_k\right)u_i \quad (124)$$

from which we have

$$u_{i,j} = \frac{w_{i,j}}{c+\beta_k x_k} - \frac{w_i\beta_j}{\left(c+\beta_k x_k\right)^2} \quad (125)$$

and then the stress component can be given by

$$\begin{aligned}\sigma_{ij} &= \left(c+\beta_l x_l\right)\left[\lambda_0\delta_{ij}w_{k,k} + \mu_0\left(w_{i,j} + w_{j,i}\right)\right]\\ &-\left[\lambda_0\delta_{ij}w_k\beta_k + \mu_0\left(w_i\beta_j + w_j\beta_i\right)\right]\end{aligned} \quad (126)$$

Substituting Eqs. (124) and (121) into Eq. (123), we obtain

$$2\mu_0 w_{j,ji} + \mu^0 w_{i,jj} + \frac{1}{\left(c+\beta_k x_k\right)}\delta\left(\mathbf{x}-\mathbf{x}^s\right)e_i = 0 \quad (127)$$

If the concentrated force acts at the origin, using the logarithmic potential, Yuan and Yin [91] obtain

$$w_{li}^{*}=\frac{1}{6\pi\mu_0}\left\{-2\delta_{li}\ln r+\frac{x_i x_l}{r^2}\right\} \tag{128}$$

where $w_{li}^{*}$ denote the generalized displacement solutions at the field point $\mathbf{x}$ along the i-direction when a unit point force is applied at the origin along the l-direction. After this, with the inverse transformation of Eq. (124), and at the same time moving the point force from the origin to an arbitrary source point $\mathbf{x}_s$, the displacement components can be written as

$$u_{li}^{*}=\frac{c}{6\pi\mu_0\left(c+\beta_k x_k\right)\left(c+\beta_k x_k^s\right)}\left\{-2\delta_{li}\ln r+\frac{r_i r_l}{r^2}\right\} \tag{129}$$

where some useful quantities related to the distance $r$ are

$$\begin{aligned} &r=\left(r_i r_i\right)^{1/2},\quad r_i=x_i-x_i^s \\ &r_{,i}=\frac{r_i}{r},\quad r_{,i}r_{,i}=1,\quad r_{i,j}=\delta_{ij} \end{aligned} \tag{130}$$

Based on the displacement formulation (129) [91], we obtain the strain components by differentiating the solution (129) with respect to the spatial variable $x_j$

$$\begin{aligned} \varepsilon_{lij}^{*}=&-\frac{c}{12\pi\mu_0\left(c+\beta_k x_k\right)^2\left(c+\beta_k x_k^s\right)} \\ &\left\{\beta_j\left(-2\delta_{il}\ln r+\frac{r_i r_l}{r^2}\right)+\beta_i\left(-2\delta_{jl}\ln r+\frac{r_j r_l}{r^2}\right)\right\} \\ &+\frac{c}{12\pi\mu_0\left(c+\beta_k x_k\right)\left(c+\beta_k x_k^s\right)}\left\{\frac{2\delta_{ij}r_l-\delta_{il}r_j-\delta_{jl}r_i}{r^2}-\frac{4r_i r_l r_j}{r^4}\right\} \end{aligned} \tag{131}$$

and then, the stress components are given by

$$\begin{aligned} \sigma_{lij}^{*}=&\frac{c}{6\pi\left(c+\beta_k x_k^s\right)}\left\{\left[\left(2\beta_l\ln r-\frac{\beta_k r_k r_l}{r^2}\right)-\left(c+\beta_k x_k\right)\frac{r_l}{r^2}\right]\delta_{ij}\right. \\ &+\beta_j\left(2\delta_{il}\ln r-\frac{r_i r_l}{r^2}\right)+\beta_i\left(2\delta_{jl}\ln r-\frac{r_j r_l}{r^2}\right) \\ &\left.+\left(c+\beta_k x_k\right)\left(\frac{2\delta_{ij}r_l-\delta_{il}r_j-\delta_{jl}r_i}{r^2}-\frac{4r_i r_l r_j}{r^4}\right)\right\} \end{aligned} \tag{132}$$

It is obvious that the fundamental solutions (129) and (132) can easily be reduced to the homogeneous fundamental solutions, when the graded parameters $\beta_i=0\ (i=1,2)$ and $c=1$. For example, for homogeneous isotropic materials with Poisson ratio $\nu=0.25$, $\lambda=\lambda_0=\mu=\mu_0$, we have

$$u_{li}^{*}=\frac{1}{6\pi\mu_0}\left[-2\delta_{il}\ln r+\frac{r_l r_i}{r^2}\right] \tag{133}$$

and

$$\sigma_{lij}^{*}=\frac{1}{6\pi}\left[\frac{r_l\delta_{ij}-r_j\delta_{il}-r_i\delta_{jl}}{r^2}-\frac{4r_i r_j r_l}{r^4}\right] \tag{134}$$

which are same as the formulations used in BEM for homogeneous materials.

## V.2 Hybrid finite element formulation

V.2.1 Hybrid functional and element stiffness equation

The initial concept of the hybrid finite element method features two independent fields (interior and frame fields) assumed in a given element. In the present work, the variables $u_i$ and $\tilde{u}_i$ respectively represent the interior and frame field variables. In the absence of body forces, the variational functional for any given element, say element e, used in the present model can be constructed as[7]

$$\Pi_{me}=\frac{1}{2}\int_{\Omega_e}\sigma_{ij}\varepsilon_{ij}\mathrm{d}\Omega-\int_{\Gamma_{te}}\bar{t}_i\tilde{u}_i\mathrm{d}\Gamma+\int_{\Gamma_e}t_i\left(\tilde{u}_i-u_i\right)\mathrm{d}\Gamma \tag{135}$$

where $\Omega_e$ is the domain of element e, $\Gamma_{te}$ and $\Gamma_{ue}$ are boundaries, where the traction and displacement are respectively specified, and $\Gamma_e$ denotes the whole element boundary. The inter-element boundary is denoted by $\Gamma_{Ie}$. Clearly, for the hybrid element shown in Fig. 2 we have

$$\Gamma_e=\Gamma_{ue}\cup\Gamma_{te}\cup\Gamma_{Ie} \tag{136}$$

Making use of Gauss theorem, the first-order variational of the functional can be further written as

$$\begin{aligned} \delta\Pi_{me}=&-\int_{\Omega_e}\sigma_{ij,j}\delta u_i\mathrm{d}\Omega+\int_{\Gamma_{Ie}}t_i\delta\tilde{u}_i\mathrm{d}\Gamma \\ &+\int_{\Gamma_{te}}\left(t_i-\bar{t}_i\right)\delta\tilde{u}_i\mathrm{d}\Gamma+\int_{\Gamma_e}\delta t_i\left(\tilde{u}_i-u_i\right)\mathrm{d}\Gamma \end{aligned} \tag{137}$$

in which the first integral gives the equilibrium equation $\sigma_{ij,j}=0$, and the second integral enforces the reciprocity condition by co-considering those from neighboring elements. The traction boundary condition can be enforced by the third integral, and the final integral enforces equality of $u_i$ and $\tilde{u}_i$ along the elemental frame boundary $\Gamma_e$.

In the present hybrid formulation, in order to obtain the element stiffness equation involving element boundary integrals only, the element interior displacement field is approximated by the linear combination of the fundamental solutions at a series of source points $\mathbf{x}_m^s$ located outside the element domain as

$$u_i(\mathbf{x}) = c_{lm}u_{li}^*(\mathbf{x},\mathbf{x}_m^s) = [\mathbf{N}_i]\{\mathbf{c}_e\}, i,l=1,2,\quad m=1,2,\cdots,M \tag{138}$$

where $M$ is the number of virtual sources outside the element domain, $\{\mathbf{c}_e\} = [c_{11} \quad c_{21} \; \vdots \; \cdots \; \vdots \; c_{1M} \quad c_{2M}]^{\mathrm{T}}$ is an unknown coefficient vector (not nodal displacement), and the interpolation matrix

$$[\mathbf{N}_i] = \left[u_{1i}^*(\mathbf{x},\mathbf{x}_1^s) \quad u_{2i}^*(\mathbf{x},\mathbf{x}_1^s) \; \vdots \; \cdots \; \vdots \; u_{1i}^*(\mathbf{x},\mathbf{x}_M^s) \quad u_{2i}^*(\mathbf{x},\mathbf{x}_M^s)\right] \tag{139}$$

consists of the fundamental solutions $u_{li}^*(\mathbf{x},\mathbf{x}^s)$ at M source points.

It is noted that the constructed displacement field (138) can analytically satisfy the inhomogeneous elastic governing equation (123), since the fundamental solutions (129) of the problem are used as the interpolation functions.

Making use of the strain-displacement equation (117) and the stress-strain relationship (112), the corresponding stress and traction components are expressed as

$$\sigma_{ij}(\mathbf{x}) = c_{lm}\sigma_{lij}^*(\mathbf{x},\mathbf{x}_m^s) = [\mathbf{S}_{ij}]\{\mathbf{c}_e\}, i,j,l=1,2,\quad m=1,2,\cdots,M \tag{140}$$

and

$$t_i(\mathbf{x}) = c_{lm}t_{li}^*(\mathbf{x},\mathbf{x}_m^s) = [\mathbf{Q}_i]\{\mathbf{c}_e\}, i,l=1,2,\quad m=1,2,\cdots,M \tag{141}$$

in which

$$[\mathbf{S}_{ij}] = \left[\sigma_{1ij}^*(\mathbf{x},\mathbf{x}_1^s) \quad \sigma_{2ij}^*(\mathbf{x},\mathbf{x}_1^s) \; \vdots \; \cdots \; \vdots \; \sigma_{1ij}^*(\mathbf{x},\mathbf{x}_M^s) \quad \sigma_{2ij}^*(\mathbf{x},\mathbf{x}_M^s)\right] \tag{142}$$

$$[\mathbf{Q}_i] = \left[t_{1i}^*(\mathbf{x},\mathbf{x}_1^s) \quad t_{2i}^*(\mathbf{x},\mathbf{x}_1^s) \; \vdots \; \cdots \; \vdots \; t_{1i}^*(\mathbf{x},\mathbf{x}_M^s) \quad t_{2i}^*(\mathbf{x},\mathbf{x}_M^s)\right] \tag{143}$$

with the traction kernels being defined by

$$t_{li}^*(\mathbf{x},\mathbf{x}_1^s) = \sigma_{lij}^*(\mathbf{x},\mathbf{x}_m^s)n_j \tag{144}$$

To enforce conformity of the displacement field on the common interface of any two neighboring elements, frame displacement fields $\tilde{u}_i$ are separately assumed on the element boundary as

$$\tilde{u}_i(\mathbf{x}) = d_{lk}\tilde{N}_{li}(\mathbf{x},\mathbf{x}_k) = [\tilde{\mathbf{N}}_i]\{\mathbf{d}_e\}, i,l=1,2,\quad k=1,2,\cdots,K \tag{145}$$

where $[\tilde{\mathbf{N}}_i]$ denotes the interpolation vector relating the boundary displacement to the nodal displacement vector $\{\mathbf{d}_e\}$.

To obtain the element stiffness equation and the optional relationship of unknown coefficient $\{\mathbf{c}_e\}$ and $\{\mathbf{d}_e\}$, the application of Gauss theorem to the functional (135) gives

$$\begin{aligned}\Pi_{me} &= -\frac{1}{2}\int_{\Omega_e}\sigma_{ij,j}u_i\mathrm{d}\Omega - \frac{1}{2}\int_{\Gamma_e}t_iu_i\mathrm{d}\Gamma \\ &\quad -\int_{\Gamma_{te}}\bar{t}_i\tilde{u}_i\mathrm{d}\Gamma + \int_{\Gamma_e}s_i\tilde{u}_i\mathrm{d}\Gamma\end{aligned} \tag{146}$$

Because the assumed displacement field (138) and stress field (140) analytically satisfy the governing equation (123), we have

$$\Pi_{me} = -\frac{1}{2}\int_{\Gamma_e}t_iu_i\mathrm{d}\Gamma - \int_{\Gamma_{te}}\bar{t}_i\tilde{u}_i\mathrm{d}\Gamma + \int_{\Gamma_e}t_i\tilde{u}_i\mathrm{d}\Gamma \tag{147}$$

Substituting Eqs. (138), (141) and (145) into the functional (147) yields

$$\Pi_{me} = -\frac{1}{2}\{\mathbf{c}_e\}^{\mathrm{T}}[\mathbf{H}_e]\{\mathbf{c}_e\} - \{\mathbf{d}_e\}^{\mathrm{T}}\{\mathbf{g}_e\} + \{\mathbf{c}_e\}^{\mathrm{T}}[\mathbf{G}_e]\{\mathbf{d}_e\} \tag{148}$$

where

$$\begin{aligned}[\mathbf{H}_e] &= \int_{\Gamma_e}[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{N}_i]\mathrm{d}\Gamma \\ [\mathbf{G}_e] &= \int_{\Gamma_e}[\mathbf{Q}_i]^{\mathrm{T}}[\tilde{\mathbf{N}}_i]\mathrm{d}\Gamma \\ \{\mathbf{g}_e\} &= \int_{\Gamma_{te}}\bar{t}_i[\tilde{\mathbf{N}}_i]^{\mathrm{T}}\mathrm{d}\Gamma\end{aligned} \tag{149}$$

The stationary condition of the functional (148) with respect to $\{\mathbf{c}_e\}$ and $\{\mathbf{d}_e\}$ yields, respectively, the optional relationship between $\{\mathbf{c}_e\}$ and $\{\mathbf{d}_e\}$ and the element stiffness equation as

$$\{\mathbf{c}_e\} = [\mathbf{H}_e]^{-1}[\mathbf{G}_e]\{\mathbf{d}_e\} \tag{150}$$

and

$$[\mathbf{K}_e]\{\mathbf{d}_e\} = \{\mathbf{g}_e\} \tag{151}$$

with

$$[\mathbf{K}_e] = [\mathbf{G}_e]^{\mathbf{T}}[\mathbf{H}_e]^{-1}[\mathbf{G}_e] \tag{152}$$

being the element stiffness matrix, which is sparse and symmetric.

## VI. CONCLUSIONS

On the basis of the preceding discussion, the following conclusions can be drawn. This review reports recent developments on applications of HFS-FEM to functionally graded materials and structures. It proved to be a powerful computational tool in modeling materials and structures with inhomogeneous properties. However, there are still many possible extensions and areas in need of further development in the future. Among those developments one could list the following:

1 Development of efficient F-Trefftz FE-BEM schemes for complex engineering structures containing heterogeneous materials and the related general purpose computer codes with preprocessing and postprocessing capabilities.

2 Generation of various special-purpose elements to effectively handle singularities attributable to local geometrical or load effects (holes, cracks, inclusions, interface, corner and load singularities). The special-purpose functions warrant that excellent results are obtained at minimal computational cost and without local mesh refinement.

3 Development of F-Trefftz FE in conjunction with a topology optimization scheme to contribute to microstructure design.

4 Extension of the F-Trefftz FEM to elastodynamics and fracture mechanics of FGMs.

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