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Wiener Index and Maximum Degree Energy of Euler Graphs

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ABSTRACT

In this paper, we study the concept of Wiener index of Euler graphs. We also introduced the concepts of Maximum degree matrix, M(G) for Euler graphs with their Maximum degree energy, $E_M(G)$. We derived the conditions for the graphs to be Eulerian. Applications of Wiener index and Euler graphs are also discussed.
Keywords : Wiener Index, Eulerian Graph, Maximum Degree Energy, Maximum Degree Graph.

I. INTRODUCTION

In this paper, all the graphs are assumed to be finite and connected. Let G be a connected graph with the vertex set $V(G)=\{v_1,v_2,\ldots,v_n\}$ and edge set $E(G)=\{e_1,e_2,\ldots,e_n\}$.

In chemical graphs, the vertices of the graph corresponds to the atoms of the molecules, and the edges represents the chemical bonds.

The study of Wiener index is one of the current area of research in mathematical chemistry. The Wiener index is one of the oldest molecular graph based structure descriptors. It was first proposed by American chemist Harold Wiener in 1947 as an aid to determining the boiling point of paraffin.[8][9]

The Wiener index of a connected graph G is defined as the half sum of the distance between all pairs of vertices of G.

It is denoted by W(G)

$W(G)=\frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$ or

$W(G)=\sum_{i<j} d(u_i,u_j)$

Where, the distance d(u,v) between the vertices u and v of the graph G is, equal to length of the shortest path that connects u and v. [9]

Our notation is standard and mainly taken from standard books of graph theory.[3,4]

II. PRELIMINARIES

2.1 Euler graph. [3,4]

A trail that traverses every edge of a graph G is called an Euler trail of G,becauseEuler (1736) was the first to investigate the existence of such trail.

A tour of a connected graph G is a closed walk that traverses each edge of G atleast once.

An Euler tour, is a tour which traverses each edge of G,exactly once.(in other words a closed Euler trail)

A graph G is Eulerian ,if it contains an Euler tour.

Note:

- ✓ Euler trail is called semi-Euler graph.
- ✓ A digraph is Eulerian, if it admits a directed Euler tour.

2.2 Examples:

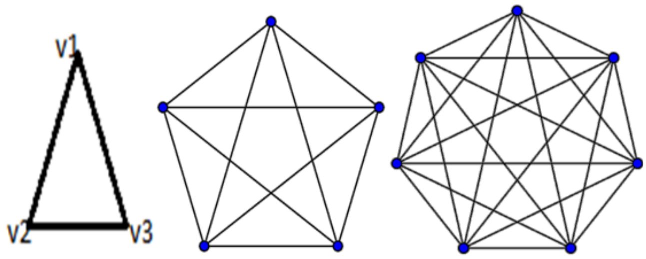


Figure 1. K_3, K_5, K_7

III. THEOREM AND RESULTS BASED ON EULER GRAPHS.

Theorem 3.1:

The following statements are equivalent for a connected graph G

1) G is Eulerian
2) Every point of G has even degree.
3) The set of lines of G can be partitioned into even cycles.

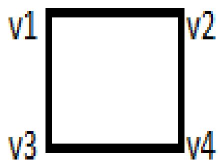
Proof:

(1) **Implies (2):** let T be an Eulerian trail in G. Each occurrence of a given point in T contributes 2 to the degree of that point, and since each line of G appears exactly once in T, every point must have even degree.

(2) **Implies (3):** Since G is connected and non-trivial, every point has degree at least 2,so G contains a cycle Z.The removal of all the lines of Z results in a spanning subgraph G_1 in which every point still has even degree. If G_1 has no lines,then (3) already holds; otherwise,a repetition of the argument applied to G_1 results in a graph G_2 in which again all points are even,etc.when a totally disconnected graph $G_n$is obtained ,we have a partition of the lines of G into n cycles.

(3) **Implies (1):** Let Z_1 be one of the cycles of partition. if G consists only of this cycle ,then G is obviously Eulerian. Otherwise, there is another cycle Z_2 with a point v in common with Z_1 .The walk beginning at v and consisting of the cycles $Z_1$and Z_2 in succession is a closed trail containing the lines of these two cycles.by continuing this process ,we can construct a closed trail containing all lines of G ;hence G is Eulerian.

Example:



Therefore A non-empty connected graph is Eulerian ,iff it has no vertices of odd degree.[3]

Corollary 3.1.1:

A connected graph has an Eulertrail,iff it has atmost two vertices of odd degree.[3,4]

Theorem 3.2:

A complete graph K_{2n+1} is Eulerian graph.

Let the given graph be a complete graph of odd number of vertices.

We know that a complete graph K_n of n vertices has n-1 degree.it follows that a graph with 2n+1 vertices has 2n degree ,which is even degree.

Therefore from theorem 3.1 and from examples 2.2 it is clear that K_{2n+1} is Eulerian graph.

Theorem 3.3:

A connected digraph is Eulerianiff for each vertex v of a connected digraph, outdeg(v) = indeg(v). [4]

IV. WIENER INDEX OF EULER GRAPH

Wiener index of Eulergraph(connected graph) is defined as the sum of the distance between all pairs of vertices of G.

W(G)=W(G)=$\sum_{i<j} \quad d(u_i, u_j)$

The Wiener index of a complete graph is defined as

W(K_n)= $\frac{n(n-1)}{2}$ n=2,3,.

From theorem 3.3 the complete graph K_{2n+1} is Euler graph.

Theorem 4.1:

The Wiener index of Euler graph K_{2n+1} is $\frac{(2n+1)(2n)}{2}$.

We prove the result by mathematical induction.

Assume that the theorem holds for $2n+1 = k \geq 1$

$W(K_k) = \frac{k(k-1)}{2}$ holds true .where $k = 2n+1$

Let v_{k+1} be an edge ordered to each other vertices by adjoining it to k-edges along with $\frac{k(k-1)}{2}$ edges.

Therefore,

$W(K_{k+1}) = k+\frac{k(k-1)}{2}$.

$= \frac{2k+k^2-k}{2}$

$=\frac{k+k^2}{2}$

$W(K_{k+1}) = \frac{k(k+1)}{2}$ where $2n + 1 = k \geq 1$

Therefore, the result holds for $n = k+1 \geq 1$

Therefore the theorem holds for all natural number n.

Hence $W(K_{2n+2}) = \frac{(2n+1)(2n+2)}{2}$ for $n \geq 1$.

Hence proved.

Example:

The Wiener index of Euler graph K_3 is 3

The Wiener index of Euler graph K_5 is 10.

V. ENERGY AND MAXIMUM DEGREE ENERGY OF THE GRAPH

Let G be a graph with n vertices and m edges . let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigen values of the adjacency matrix of G.

The energy of G was first defined by Gutman [6] in 1978 as the sum of the absolute values of its eigenvalues.

The energy, E(G), of a graph G is defined to be the sum of the absolute values of its eigenvalues. Hence if A(G) is the adjacency matrix of G and $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigen values of the adjacency matrix of G.

Then $E(G) = \sum_{i=1}^{n} |\lambda_i|$

The set $\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ is the spectrum of G. [7]

During these days the energy of a graph is a much studied quantity in the mathematical literature. Motivated by recent works on energy of a graph, in this paper we introduce maximum degree matrix associated with a Euler graph.

Let G be a simple graph with n vertices $v_1,v_2,\ldots,v_n$ and let d_i be the degree of v_i .i=1,2,3,..n

[5] Define,

$$d_{ij}= \begin{cases} \max\{d_i, d_j\} \text{ if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 \text{ otherwise} \end{cases}$$

then the n x n matrix $M(G) = [d_{ij}]$ is called the maximum degree matrix of G. The characteristic polynomial of the maximum degree matrix M(G) is defined by,

$$\phi(G;\mu) = \det(\mu I - M(G))$$
$$=\mu^n + c_1\mu^{n-1} + c_2\mu^{n-2} + \cdots + c_n.$$

Where I is the unit matrix of order n. The roots $\mu_1, \mu_2, \ldots, \mu_n$ assumed in non-increasing order of $\phi(G;\mu) = 0$ are the maximum degree eigenvalue of G.

The maximum degree energy of a graph G is defined as

$E_M(G) = \sum_{i=1}^{n} |\mu_i|$

Since M(G) is real symmetric matrix with zero trace, these maximum degree eigen values are real with sum equal to zero. Thus $\mu_1, \geq \mu_2 \geq \ldots \geq \mu_n$ and

$\mu_1 + \mu_2 + \ldots + \mu_n = 0$.

Example 5.1: To find $E_M(K_3)$.

$$M(K_3) = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}$$

The characteristics polynomial of the $M(K_3)$ is $\phi(K_3;\mu)=\det(\mu I-M(K_3))$

$$= \begin{vmatrix} \mu & -2 & -2 \\ -2 & \mu & -2 \\ -2 & -2 & \mu \end{vmatrix}$$

$= \mu(\mu^2-4) + 2(-2\mu-4) - 2(4+2\mu)$

$= \mu^3-12\mu - 16$

And the maximum degree eigen values of K_3 are
$\mu_1 = 4$ $\mu_2 = -2$ $\mu_3 = -2$

hence, $E_M(K_3) = 4+2+2 = 8$.

Theorem 5.1:
If G is the complete graph K_n, where n is odd number of vertices, then $-(n-1)$ and $-(n-1)^2$ are the maximum degree eigenvalues of G with multiplicity (n-1) and 1 respectively and $E_M(K_n) = 2(n-1)^2$,where K_n is Euler graph.

Proof: $\phi(G;\mu) = \det(\mu I - M(G))$

$$\det(\mu I - M(\mathrm{Kn}))=\begin{vmatrix} \mu & -(n-1) & -(n-1) & \dots & -(n-1) \\ -(n-1) & \mu & -(n-1) & \dots & -(n-1) \\ \dots. & \dots & & \dots. & \\ -(n-1) & -(n-1) & -(n-1) & & \mu \end{vmatrix}$$

$$= (\mu+(n-1))^{n-1}\begin{vmatrix} \mu & -(n-1) & -(n-1) & \dots & -(n-1) \\ -1 & 1 & 0 & \dots & 0 \\ \dots. & \dots & & \dots. & \\ -1 & 0 & 0 & & 1 \end{vmatrix}$$

$= (\mu+(n-1))^{n-1} (\mu-(n-1))^2$

Hence , $E_M(K_n)=2(n-1)^2$.

Result: If the maximum degree energy of a graph is rational, then it must be an integer.[5]

VI. APPLICATIONS

One of the main applications of graph spectra to chemistry is the application in a theory of unsatured conjugated hydrocarbons knowns as the Hückel molecular orbital theory. Apart from chemistry there are many interactions between the theory of graph spectra and other branches of mathematics,like linear algebra and in combinatorial matrix theory.

Applications of Euler graphs are started from the problem of Konigsberg seven bridge to the current problrm of DNA fragement assembly. [2].

VII. CONCLUSION

In this paper , we discussed Wiener index and maximum degree energy of Euler graphs with their applications.

This article is intended for the attention of young readers, uninitiated in graph theory and gives an discussion of certain well-known Euler graphs with their properties.

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