A Study on Prime Labeling of Some Special Graphs

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ABSTRACT

In this paper, we discussed about the prime labeling for Herschel graph and cubic graph with 8 vertices. A graph G with vertex set V is said to have a prime labeling, if its vertices are labeled with integers 1,2,3,...,[V]. Such that for each xy the labels assigned to x and y are relatively prime. A graph which admits prime labeling is called a prime graph. In this paper we also discuss prime labeling in the context of some graph operations namely Fusion, Duplication and Switching.

Keywords: Prime Labeling, Fusion, Duplication And Switching.

I. INTRODUCTION

This works deals with graph labeling. All the graphs considered here are finite and undirected. The graph G=(V(G), E(G)) has vertex set V=V(G) and edge set E=E(G). The set of vertices adjacent to a vertex u of G is denoted by N(u).

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is set of vertices(or edges) then the labeling is called a vertex labeling(or an edge labeling).

Following are the common features of any graph labeling problem

✓ A set of numbers from which vertex labels are assigned.
✓ A rule that assigns value to each edge.
✓ A condition that these values must satisfy.

The notation of prime labeling was introduced by Roger Entringer and was discussed in a paper by A. Tout (1982 P 365-368) [9]. Many researches have studied prime graph for example in H.C. Fu (1994 P 181-186) [1] have proved that path P_n on n vertices is a prime graph.

T.O Dertsky (1991 P 359-369) [8] have proved that the cycle C_n on n vertices is a prime graph. S.M. Lee (1998 P 59-67) [7] have proved that Wheel W_n is a prime graph iff n is even . Around 1980 Roger Entringer conjectured that all tress have prime labelling, which is not settled till today.


We will provide brief summary of definitions and some other information which are necessary for the present investigations.

Definition : 1

If the vertices of the graph are assigned values subject to certain conditions then it is called as (vertex) graph labeling.

Definition : 2

Let G(V, E) be a graph with n vertices. A bijection f: V → {1,2,3,...,n} is called a prime labeling if for
A graph which admits prime labeling is called a prime graph.

**Definition : 3**
An independent set of vertices in a graph $G$ is a set of mutually non-adjacent vertices.

**Definition : 4**
Let $u$ and $v$ be two distinct vertices of a graph $G$. A new graph $G_1$ is constructed by fusing (identifying) two vertices $u$ and $v$ by a single vertex $x$ in $G_1$ such that every edge which was incident with either $u$ (or) $v$ in $G$ now incident with $x$ in $G_1$.

**Definition: 5**
Duplication of a vertex $v_k$ of a graph $G$ produces a new graph $G_1$ by adding a vertex $v'_k$ with $N(v'_k) = N(v_k)$. In other words, a vertex $v'_k$ is said to be a duplication of $v_k$ if all the vertices which are adjacent to $v_k$ are now adjacent to $v'_k$.

**Definition: 6**
A vertex switching of a graph $G$ is obtained by taking a vertex $v$ of $G$, removing the entire edges incident with $v$ and adding edges joining $v$ to every vertex which are not adjacent to $v$ in $G$.

**Definition :6**
A bipartite undirected graph with 11 vertices and 18 edges is called as Herschel Graph, it is denoted as $H_S$.

**Theorem 1:**
The Herschel graph $H_S$ is a prime graph.

**Proof:**
First let us assume that $H_S$ be the Herschel graph with 11 vertices and 18 edges. Assume that ‘c’ be the centre of the Herschel graph.

Then $|V(H_S)| = 11$ and $|E(H_S)| = 18$

Consider, $f(c) = 1$ and $f(u_i) = 2i$ for $1 \leq i \leq 4$, where $u_i’s$ are adjacent to $c$.

Therefore,

- $f(u_1) = 2$
- $f(u_2) = 4$
- $f(u_3) = 6$
- $f(u_4) = 8$

After that we assign the label vertex is $u_5$, but $u_5$ is adjacent to $u_1$ and $u_4$.

And the vertices $u_1$ and $u_4$ are having even label. Therefore, $f(u_5) = 3$

Next we assign the label of vertex is $u_6$, then $u_6$ is adjacent to $u_2$ and $u_3$.

And the vertices $u_2$ and $u_3$ are having even label. Therefore, $f(u_6) = 5$

In the same way, $u_7$ is adjacent to $u_3$ and $u_4$ also having even label.

Therefore, $f(u_7) = 7$

And $u_8$ is adjacent to $u_1$ and $u_2$ having even label. Therefore, $f(u_8) = 9$

Finally, Let $f(u_9) = 10$ and $f(u_{10}) = 11$

Thus, for every edge $e = cu_i \in H_S$, $\gcd(f(c), f(u_i)) = 1$ and the edge $e = u_iu_j \in H_S$, $\gcd(f(u_i), f(u_j)) = 1$.

Then Herschel graph $H_S$ admits prime labeling. Hence Herschel graph $H_S$ is a prime graph.

**Example**
Let \( f(c) = 1 \) and \( f(u_i) = 2i \) for \( 1 \leq i \leq 4 \), where \( u_i \)'s are adjacent to \( c \).
Therefore,
\[
\begin{align*}
  f(u_1) &= 2 \\
  f(u_2) &= 4 \\
  f(u_3) &= 6 \\
  f(u_4) &= 8
\end{align*}
\]
After that we assign the label is \( u_2 \) but it is adjacent to \( u_1 \) and \( u_4 \) and the vertices having even label.
Therefore \( f(u_5) = 3 \)

In the same way, \( u_6 \) is adjacent to \( u_2 \) and \( u_3 \) also having even label.
Therefore, \( f(u_6) = 5 \)

In the same way, we assign the label of the vertex is \( u_7 \) and \( u_8 \).
Therefore, Let \( f(u_7) = 7 \) and \( f(u_8) = 9 \)
Finally, we assign the label of the vertex is \( u_9 \) with the remaining label is 10,
Therefore, \( f(u_9) = 10 \)

Now, for each edge \( e = cu_i \in H_5, \text{gcd}(f(c), f(u_i)) = 1 \) and the edge \( e = u_iu_j \in H_5, \text{gcd}(f(u_i), f(u_j)) = 1 \).

Then \( G \) admits prime labeling.
Hence \( G \) is a prime graph.

**Example**

**Theorem 2:**
The fusion of two adjacent vertices of degree 3 in a Herschel graph is a prime graph.

**Proof:**
First we suppose that \( H_5 \) be the Herschel graph with 11 vertices and 18 edges.

Next we assume that \( c \) be the centre of the Herschel graph \( H_5 \) and it has 8 vertices of degree 3 and 3 vertices of degree 4.

Then \( |V(H_5)| = 11 \) and \( |E(H_5)| = 18 \)

Consider \( G \) be the graph obtained by fusing of two adjacent vertices of degree 3 in the Herschel graph \( H_5 \).
Therefore \( |V(H_5)| = 10 \)

Now we define a bijective function \( f: V(G) \rightarrow \{1,2,3,4, ..., 10\} \)
Theorem 2:
The fusion of two adjacent vertices of degree 3 in a Herschel graph is a prime graph.

Proof:
First we suppose that $H_s$ be the Herschel graph with 11 vertices and 18 edges.

Next we assume that $c$ be the centre of the Herschel of graph $H_s$ and it has 8 vertices of degree 3 and 3 vertices of degree 4.

Then $|V(H_s)| = 11$ and $|E(H_s)| = 18$

Consider $G$ be the graph obtained by fusing of two adjacent vertices of degree 3 in the Herschel graph $H_s$.

Therefore $|V(H_s)| = 10$

Now we define a bijective function $f: V(G) \rightarrow \{1,2,3,4, \ldots, 10\}$

Let $f(c) = 1$ and $f(u_i) = 2i$ for $1 \leq i \leq 4$, where $u_i$'s are adjacent to $c$.

Therefore, 
\[
\begin{align*}
    f(u_1) &= 2 \\
    f(u_2) &= 4 \\
    f(u_3) &= 6 \\
    f(u_4) &= 8 \\
\end{align*}
\]

After that we assign the label is $u_5$ but it is adjacent to $u_1$ and $u_4$ and the vertices having even label.

Therefore $f(u_5) = 3$

In the same way, $u_6$ is adjacent to $u_2$ and $u_3$ also having even label.

Therefore, $f(u_6) = 5$

In the same way, we assign the label of the vertex is $u_7$ and $u_8$.

Therefore, Let $f(u_7) = 7$ and $f(u_8) = 9$

Finally, we assign the label of the vertex is $u_9$ with the remaining label is 10.

Therefore, $f(u_9) = 10$

Now, for each edge $e = cu_i \in H_s$, $gcd(f(c), f(u_i)) = 1$ and the edge $e = u_iu_j \in H_s$, $gcd(f(u_i), f(u_j)) = 1$.

Then $G$ admits prime labeling.

Hence $G$ is a prime graph.

Example

Fig 3 Fusion of the vertices $u_6$ and $u_{10}$ in a Herschel graph $H_s$ is a prime graph.

Theorem 3:
The Duplication of any vertex of 3 in a Herschel graph is a prime graph.

Proof:

Let $H_s$ be the Herschel graph with 11 vertices and 18 edges and $c$ be the centre vertex of the graph.

Then $|V(H_s)| = 11$ and $|E(H_s)| = 18$

Let us assume that $u_k$ be the any vertex of degree 3 and $u'_k$ be the duplication of the vertex $u_k$ in the Herschel graph $H_s$.

Now we assume that $G_k$ be the graph obtained by after duplicating the vertex $u_k$ of degree 3 in Herschel graph $H_s$. So $|V(G_k)| = 12$.

Next we define the label of bijective function $f: V(G_k) \rightarrow \{1,2,3,4, \ldots, 12\}$. 

Let us assume that $f(c) = 1$. Then $f(u'_k) = 12$ where $u'_k$ is the duplicating vertex of $u_k$. And $f(u_i) = 2i$ for $1 \leq i \leq 4$, where $u'_i$'s are adjacent to $c$.

Therefore, $f(u_4) = 2$

\[
f(u_2) = 4 \\
f(u_3) = 6 \\
f(u_4) = 8
\]

Let $f(u_5) = 3$, by reason of vertex $u_5$ is adjacent to $u_1$ and $u_4$ both having an even label.

$f(u'_k) = 5$, by reason of vertex $u'_6$ is adjacent to $u_2$ and $u_3$ both having an even label.

$f(u_7) = 7$, by reason of vertex $u_7$ is adjacent to $u_3, u_4, u_9, u_{10}$ and $u_k$.

$f(u_8) = 11$, by reason of vertex $u_8$ is adjacent to $u_1, u_2, u_9, u_{10}$ and $u_k$.

$f(u_9) = 10$, by reason of vertex $u_9$ is adjacent to $u_5, u_7, u_8$ are having even label.

$f(u'_k) = 9$. Here we duplicating $u'_k$ in Herschel graph $H_5$ as $u'_k$.

Now, for every edge $e = cu_i \in G_k$, $\gcd(f(c), f(u_i)) = 1$ and the edge $e = u_iu_j \in G_k$, $\gcd(f(u_i), f(u_j)) = 1$.

Then $G_k$ admits prime labeling.

Hence $G_k$ is a prime graph.

**Example**

![Fig 4 Duplication of the vertex $u_{10}$ of degree 3 in $H_5$ is a prime graph](image)

**Theorem 4:**
Switching the centre vertex $c$ in the Herschel graph $H_5$ is a prime graph.

**Proof:**
Consider $H_5$ be the Herschel graph with 11 vertices and 18 edges and $c$ be the centre vertex of the graph.

Then $|V(H_5)| = 11$ and $|E(H_5)| = 18$

Let us assume that $c$ be the switching vertex and $G_c$ be the new graph obtained by switching the centre vertex $c$.

Clearly $|V(G_c)| = 11$ and $|E(G_c)| = 19$

Next we define the label of bijective function $f: V(G_c) \rightarrow \{1, 2, 3, 4, \ldots, 11\}$.

Such that $f(c) = 1$. And $f(u_i) = 2i$ for $1 \leq i \leq 4$, where $u'_i$'s are adjacent to $c$.

Therefore, $f(u_4) = 2$

\[
f(u_2) = 4 \\
f(u_3) = 6 \\
f(u_4) = 8
\]

Now we assume that $f(u_5) = 3$, since vertex $u_5$ is adjacent to $u_4$ and $u_4$ both having an even labels.

$f(u'_6) = 5$, since vertex $u_6$ is adjacent to $u_2$ and $u_3$ of even labels.

$f(u_7) = 7$, since vertex $u_7$ is adjacent to $u_3$ and $u_4$ of even labels

$f(u_8) = 9$, since vertex $u_9$ is adjacent to $u_1$ and $u_2$ having an even labels.

$f(u_9) = 10$, since $u_9$ is adjacent to $u_7$ and $u_8$ having an even labels

Finally, $f(u_{10}) = 11$

For each edge $e = u_iu_j \in G_c$, $\gcd(f(u_i), f(u_j)) = 1$.

Then $G_c$ admits a prime labeling.

Hence $G_c$ is a prime graph.
ON PRIME LABELING OF CUBIC GRAPH WITH 8 VERTICES

Cubic Graph

A Regular graph $G$ is called a Cubic graph if all the vertices of $G$ are of degree 3.

Theorem 1:
A cubic graph with 8 vertices is a prime graph.

Proof:
Let $G = (V, E)$ be a cubic graph with 8 vertices and 12 edges.

Now we assume that the vertex set $V(G) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and the edge set $E(G) = \{u_iv_i / 1 \leq i \leq 4\} \cup \{u_iu_{i+1} / 1 \leq i \leq 3, u_4u_1\} \cup \{v_iv_{i+1} / 1 \leq j \leq 3, v_4v_1\}$

Then $|V(G)| = 8$ and $|E(G)| = 12$

Let us suppose that define a labeling function $f: V(G) \rightarrow \{1, 2, 3, ... 8\}$ such that $f(u_i) = i$ for $1 \leq i \leq 4$

$f(u_1) = 1$
$f(u_2) = 2$
$f(u_3) = 3$
$f(u_4) = 4$

and $f(v_i) = f(u_i) + 5$ for $1 \leq i \leq 3$

(i.e.) $f(v_1) = f(u_1) + 5 = 1 + 5 = 6$

Similarly, $f(v_2) = f(u_2) + 5 = 2 + 5 = 7$

$f(v_3) = f(u_3) + 5 = 3 + 5 = 8$

Finally, $f(v_4) = f(u_4) + 1 = 4 + 1 = 5$

Thus, for each edge $e = u_iu_j \in G$, $\gcd(f(u_i), f(v_j)) = 1$ and the edges

$e = u_1u_2, v_1v_2 \in G, \gcd(f(u_1), f(v_1)) = 1$ and $\gcd(f(v_1), f(v_2)) = 1.$

Then $G$ admits a prime labeling.
Hence, $G$ is a prime graph.

Theorem 2:
The fusion of two consecutive vertices in the outer cycle graph on 8 vertices is a prime graph.
Proof:
Let us assume that $G = (V, E)$ be a cubic graph on 8 vertices and $G_f$ be the graph obtained by fusion (or identifying) two vertices $v_1$ and $v_2$ (i.e. $v_1 = v_2$) of $G$.

$(i.e., |G_f(V)| = 7$

Now we define a label function $f: V(G_f) \rightarrow \{1,2,3,...,7\}$

Such that $f(u_i) = i$ for $1 \leq i \leq 4$

\[
f(u_1) = 1
\]

\[
f(u_2) = 2
\]

\[
f(u_3) = 3
\]

\[
f(u_4) = 4
\]

And let $f(v_1) = f(u_1) + 5$ for $1 \leq i \leq 3$

$(i.e.)$

\[
f(v_1) = f(u_2) + 5 = 1 + 5 = 6
\]

Similarly, $f(v_2) = f(u_2) + 5 = 2 + 5 = 7$

Finally, $f(v_4) = f(u_4) + 1 = 4 + 1 = 5$

Clearly, for each edge $e = u_iu_j \in G$, $gcd(f(u_i), f(v_i)) = 1$

and the edges $e = u_iu_j, v_iv_j \in G, gcd \left( f(u_i), f(u_j) \right) = 1$ and $gcd \left( f(v_i), f(v_j) \right) = 1$.

Then $G_f$ admits a prime labeling.

Hence $G_f$ is a prime graph.

Example:

Fig 2: Fusion of two vertices $v_2$ and $v_3$ in a cubic graph is a prime graph.

Theorem 3:
The Duplication of an arbitrary vertex of the cubic graph on 8 vertices produces a prime graph.

Proof:
Consider $G = (V, E)$ be a cubic graph with 8 vertices and 12 edges.

Now we assume that the vertex set is $V(G) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and the edge set $E(G) = \{u_i, v_i / 1 \leq i \leq 4\} \cup \{u_iu_i+1 / 1 \leq i \leq 3, u_4, u_1\} \cup \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4, v_3v_4\}$

Let $G_k$ be the graph obtained by duplicating and arbitrary vertex of $G$. Without loss of generality, let this vertex be $v_1$ and the newly added vertex be $v_1'$.

Now we define the label function $f: V(G_k) \rightarrow \{1,2,3,...,9\}$

Such that $f(u_i) = i$ for $1 \leq i \leq 4$

\[
f(u_1) = 1
\]

\[
f(u_2) = 2
\]

\[
f(u_3) = 3
\]

\[
f(u_4) = 4
\]

And let $f(v_1) = f(u_1) + 5$ for $1 \leq i \leq 3$

$(i.e.)$

\[
f(v_1) = f(u_2) + 5 = 1 + 5 = 6
\]

Similarly, $f(v_2) = f(u_2) + 5 = 2 + 5 = 7$

Finally, $f(v_4) = f(u_4) + 1 = 4 + 1 = 5$

Thus, for each edge $e = u_iu_j, u_iu_i, v_iv_j \in G, gcd(f(u_i), f(v_i)) = 1$,

$gcd \left( f(u_i), f(u_j) \right) = 1$ and $gcd \left( f(v_i), f(v_j) \right) = 1$.

Then $G_k$ admits a prime labeling.

Hence $G_k$ is a prime graph.

Example:
Fig 3 The Duplication of the vertex \( v_1 \) in cubic graph is a prime graph

**Theorem 4:**

The Switching of an arbitrary vertex in a cubic graph on 8 vertices is a prime graph.

**Proof:**

Consider \( G = (V, E) \) be a cubic graph with 8 vertices and 12 edges.

Now we assume that the vertex set is \( V(G) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\} \) and the edge set \( E(G) = \{u_i v_i / 1 \leq i \leq 4\} \cup \{u_i u_{i+1} / 1 \leq i \leq 3, u_4, u_1\} \cup \{v_1 v_{i+1} / 1 \leq i \leq 3, v_4 v_1\} \)

Assume that \( G_s \) be the graph obtained by switching and arbitrary vertex of \( G \). Without loss of generality let this vertex be \( v_1 \) and \( |V(G)| = 8 \) and \( |E(G)| = 12 \)

Now we define the label function \( f: V(G_s) \rightarrow \{1, 2, 3, ... 8\} \)

Such that \( f(u_i) = i \) for \( 1 \leq i \leq 4 \)

\( f(v_1) = 1 \), here \( v_1 \) is a switching vertex

\( f(v_2) = 2 \)

\( f(v_3) = 3 \)

\( f(v_4) = 4 \)

(i.e.) \( f(u_1) = f(v_1) + 5 = 1 + 5 = 6 \) and \( f(v_1') = 9 \)

\( f(u_2) = f(v_2) + 5 = 2 + 5 = 7 \)

Similarly, \( f(u_3) = f(v_3) + 1 = 3 + 5 = 8 \)

Finally, \( f(u_4) = f(v_4) + 1 = 4 + 1 = 5 \)

Thus, for every edge \( e = u_i v_i \in G, \gcd(f(u_i), f(v_i)) = 1 \) and the edges \( e = u_i u_j, v_i v_j \in G, \gcd(f(u_i), f(v_j)) = 1 \) and \( \gcd(f(v_i), f(v_j)) = 1 \).

Then \( G_s \) admits a prime labeling.

Hence \( G_s \) is a prime graph.

**Example**

Fig 4 The Switching of \( v_1 \) in a cubic graph is a prime graph.

### II. CONCLUSION

In this dissertation, we have investigate a four results corresponding to prime labeling on some special graphs, namely Herschel graph and cubic graph with 8 vertices. Analogous work can be carried out for other families and in the context of different types of graph labeling techniques.
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IV. REFERENCES