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Reduction of Fractional Differential Equation (FDE) to Ordinary Differential Equation (ODE)

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ABSTRACT

In this paper, Will show that how the solution of the fractional differential equation system can be converted into a problem in ordinary differential equation in two method. With this method the only time the calculation fractional differential equation enters in to the picture is in the calculation of fractional derivatives of known functions. To reach this thing we will use the Laplace transformation in first method and the convolution of the concept of fractional green's function in the second method.
Keywords: Fractional Differential Equation, Ordinary Differential Equation

I. INTRODUCTION

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electron-analytical chemistry, biology, control theory, fitting of experimental data, and so forth, and involves derivatives of fractional order. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional differential equations in comparison with classical integer-order models.

Integer-order impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments. [2][3].

On the other hand, impulsive differential equations of fractional order play an important role in theory and applications. The fractional impulsive differential equations have not been addressed so extensively and many aspects of these problems are yet to be explored. For example, the theory using Green's function to express the solution of fractional impulsive differential equations has not been investigated till now. in this paper, we shall study the expression of the solution of fractional impulsive differential equations by using Green's function.

We need definition some special functions:

1.1 The Mittag-Leffler Function:[2][3][4]

The Mittag-Leffler function is a direct generalization of the exponential function, e^x and it plays a major role in fractional calculus. The one and two-parameter representations of the Mittag-Leffler function can be defined in terms of a power series as

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + 1)}, \alpha > 0 \qquad (1.1)$$

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)}, \alpha > 0, \beta > 0 \quad (1.2)$$

As a result of the definition given in (1.6), the following relations hold:

$$E_{\alpha,\beta}(x) = \frac{1}{\Gamma(\beta)} + x E_{\alpha,\alpha+\beta}(x). \qquad (1.3)$$

And

$$E_{\alpha,\beta}(x) = \beta E_{\alpha,\beta+1}(x) + \alpha x \frac{d}{dx} E_{\alpha,\beta+1}(x). \quad (1.4)$$

Observe that (1.8) implies that

$$\frac{d}{dx} E_{\alpha,\beta+1}(x) = \frac{1}{\alpha x}[E_{\alpha,\beta}(x) - \beta E_{\alpha,\beta+1}(x)]. \qquad (1.5)$$

So

$$\frac{d}{dx} E_{\alpha,\beta}(x) = \frac{1}{\alpha x}[E_{\alpha,\beta-1}(x) - (\beta-1)E_{\alpha,\beta}(x)]. \qquad (1.6)$$

The exponential series defined by (1.6) gives a generalization of (1.5). Note that $E_{\alpha,\beta}(0) = 1$. Also, for some specific values of and, the Mittag-Leffler function reduces to some familiar functions. For example.

$$E_{1,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x. \qquad (1.7a)$$

$$E_{1,2}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+2)} = \frac{1}{x}\sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} = \frac{e^x - 1}{x}. \quad (1.7b)$$

1.2. TheMellin-Ross Function:[2][3][4]

The Mellin-Ross function, $E_t(v,a)$ arises when finding the fractional derivative of an exponential e^{at}. The function is closely related to both the incomplete Gamma and Mittag-Leffler functions. Its definition is given by

$$E_t(v,a) = t^v \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma(k+v+1)} = t^v E_{1,v+1}(at). \qquad (1.8)$$

and the notation $E_t(v,a)$ is a solution of the ordinary differential equation $Dy - ay = \frac{t^{v-1}}{\Gamma(v)}$

Basic definitions

Definition 2.1 : [2][3][4]Let α > 0, The Riemann-Liouville left-side fractional integral I_{a+}^{α} f of order α with original at the point a is defined by:

$$I_{t+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, t > a, \ (1.9)$$

Provided the integral exists. Here Γ(α) is the Gamma function, and I_{a+}^{α} is called an integral operator of order α.

Definition 2.2:[2][3][4]Let $n-1 \le \alpha \le n, n \in N$, The Riemann -Liouville fractional derivative $D_{a+}^{\alpha} f$ of order α with original at the pointa is defined by:

$$D_{t+}^{\alpha} f(t) = D^n I_{t+}^{n-\alpha} f(t) = \left(\frac{d}{dt}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, t > a, (1.10)$$

and D_{t+}^{α} is called the fractional differential operator of order α.

Definition 2.3:[6]The Laplace Transform

We recall that a function $y(t)$ defined on some domain J' is said to be of exponential order if there exist constants N, T such that $e^{-\alpha t}|y(t)| \le N$ for all $t \ge T$. If $y(t)$is of exponential order α, then $\int_0^{\infty} y(t)e^{-st} dt$ exists for all $Re\ s > \alpha$. The Laplacetransform of $y(t)$is then defined as in [6].

$$Y(t) = \mathcal{L}\{y(t)\} = \int_0^{\infty} y(t)e^{-st} dt \, (1.11)$$

We say that $y(t) = \mathcal{L}^{-1}\{y(t)\}$is the (unique) inverse Laplace transform of $Y(t)$.

We also recall that the Laplace transform is a linear operator [1]. In particular, if $\mathcal{L}\{f(t)\}\ and\ \mathcal{L}\{g(t)\}$ exist, then

$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$ and

$\mathcal{L}\{cf(t)\} = c\mathcal{L}\{g(t)\}(1.12)$

For some elementary calculus example shows that for all $\mu > -1$ and

$a \in \mathbb{R}, \mathcal{L}\{t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+1}}, and\ \mathcal{L}\{e^{at}\}(1.13)$

One of the most useful properties of the Laplace transform is found in the convolution theorem. This theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transforms. So, if $F(s)$and $G(s)$are the Laplace transforms of $f(t)\ and\ g(t)$ and, respectively, then where:[1]

$$f * g = F(s)\, G(s),$$

Where

$f * g = \mathcal{L}\left\{\int_0^t f(t-z)g(z)dz\right\}$ (1.14)

Now we put two new methods to transform the fractional differential equation to ordinary differential equation

II. METHODS AND MATERIAL

A. The Laplace transformation method

werecall the fractional integral of $f(t)$ of order is

$$I_t^{-v}f(t) = \frac{1}{\Gamma(v)}\int_0^t (t-x)^{v-1}f(x)dx$$

This equation is actually a convolution integral. So, using (1.12) and (1.13) we findthat :

$$\mathcal{L}\{I_t^{-v}f(t)\} = \frac{1}{\Gamma(v)}\mathcal{L}\{t^{v-1}\}\mathcal{L}\{f(t)\} = s^{-v}Y(s), v > 0 \ (1.15)$$

Equation (1.15) is the Laplace transform of the fractional integral. As examples, we seefor $, v > 0 , \mu > -1$ that

$$\mathcal{L}\{I_x^{-v}t^{\mu}\} = \frac{\Gamma(\mu+1)}{s^{\mu+v+1}} \ and \ \mathcal{L}\{I_x^{-v}e^{at}\} = \frac{1}{s^v(s-a)} (1.16)$$

We again recall that in the integer order operations, the Laplace transform of y $y^{(n)}$ is given by:

$$\mathcal{L}\{y^{(n)}\} = s^nY - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{n-1}(0)$$

$$= s^nY(s) - \sum_{k=0}^{n-1} s^{n-k-1}\, y^k(0) (1.17)$$

Now from probability of fractional derivative [2] and by the fractional ofderivative $y(t)$ of order α is

$$D_x^{\alpha}y(t) = D_x^n[D_x^{-u}y(t)] (1.18a)$$

Where n, is the smallest integer greater than $\alpha > 0$, and

$$u = n - \alpha$$

we can write equation (1.18a) as:

$$D_x^{\alpha}y(t) = D_x^n\left[D_x^{-(n-\alpha)}y(t)\right] (1.18b)$$

Now, if we assume that the Laplace transform of $y(t)$ exists, then by the use of (1.15) we have:

$$\mathcal{L}\{D_x^{\alpha}y(t)\} = \mathcal{L}\left\{D_x^n\left[D_x^{-(n-\alpha)}y(t)\right]\right\}$$

$$= s^n\mathcal{L}\left\{D_x^{-(n-\alpha)}y(t)\right\} - \sum_{k=0}^{n-1} s^{n-k-1}D_x^k\left[D_x^{-(n-\alpha)}y(t)\right]_{t=0}$$

$$= s^n\left[s^{-(n-\alpha)}Y(s)\right] - \sum_{k=0}^{n-1} s^{n-k-1}\,D_x^{k-(n-\alpha)}y(0)$$

$$= s^{\alpha}Y(s) - \sum_{k=0}^{n-1} s^{n-k-1}D_x^{k-n+\alpha}y(0) (1.19)$$

In particular, if $n = 1$ and $n = 2$, we respectively have

$$\mathcal{L}\{D_x^{\alpha}y(t)\} = s^{\alpha}Y(s) - D_x^{-(1-\alpha)}y(0), 0 < \alpha \leq 1 . \quad (1.20)$$

$$\mathcal{L}\{D_x^{\alpha}y(t)\} = s^{\alpha}Y(s) - sD_x^{-(2-\alpha)}y(0) - D_x^{-(1-\alpha)}y(0), 1 < \alpha \leq 2. (1.21)$$

Table 1.1gives a brief summary of some useful Laplace transform pairs. We willfrequently refer to this Table.This function plays an important role when solving fractional differential equations.

Table 1.1. Laplace transforms pairs

$Y(s)$	y(t)
$\frac{1}{s^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}$
$\frac{1}{(s+a)^{\alpha}}$	$\frac{t^{\alpha-1}}{\Gamma(\alpha)}e^{-at}$
$\frac{1}{s^{\alpha}-a}$	$t^{\alpha-1}E_{\alpha,\alpha}(at^{\alpha})$
$\frac{s^{\alpha}}{s(s^{\alpha}-a)}$	$E_{\alpha}(-at^{\alpha})$
$\frac{a}{s(s^{\alpha}-a)}$	$1 - E_{\alpha}(-at^{\alpha})$
$\frac{1}{s^{\alpha}(s-a)}$	$t^{\alpha}E_{1,\alpha+1}(at)$
$\frac{s^{\alpha-\beta}}{s^{\alpha}-a}$	$t^{\beta-1}E_{\alpha,\beta}(at^{\alpha})$
$\frac{s^{\alpha-\beta}}{(s-a)^{\alpha}}$	$\frac{t^{\beta-1}}{\Gamma(\beta)}F_1(\alpha;\beta;at)$
$\frac{1}{(s-a)(s-b)}$	$\frac{1}{a-b}(e^{at}-e^{bt})$

In this table, a and $a \neq b$ are real constants; α, β are arbitrary.

Example 1 : Let's solve $D_t^{\frac{3}{4}}y(t)$ where a is constant.Sine $0 < \alpha = \frac{3}{4} \leq 1$, we will use (1.20). Taking the Laplace transform of both sides of the equation we have:

$$\mathcal{L}\left\{D_t^{3/4}y(t)\right\} = a\mathcal{L}\{y(t)\}$$

which implies that

$$s^{3/4}Y(s) - D_{t=0}^{-(1-3/4)}y(0) = aY(s)(1.22)$$

The constant $D_{t=0}^{-(1-3/4)}y(0) = D_{t=0}^{-1/4}y(0)$ is the value of $D_t^{-1/4}y(t) at\ t = 0$

If we assume that this value exists, and call it , then (1.22) becomes

$$s^{3/4}Y(s) - c_1 = aY(s)$$

Solving for $Y(s)$ we obtain

$$Y(s) = \frac{c_1}{s^{3/4} - a}$$

Finally, using Table 1.1 we find the inverse Laplace of $Y(s)$, and conclude that

$$y(t) = \mathcal{L}^{-1}\left\{\frac{c_1}{s^{3/4} - a}\right\} = c_1 t^{-1/4} E_{3/4,3/4}\left(at^{3/4}\right)$$

Example 2 : Let's solve $D_t^{3/2}y(t) = 0$

Sine $0 < \alpha = 3/2 \leq 2$, we will use (1.21). Taking the Laplace transform of both sides of the equation we have:

$$\mathcal{L}\left\{D_t^{3/2}y(t)\right\} = 0$$

which implies that

$$s^{3/2}Y(s) - sD_{t=0}^{-(2-3/2)}y(0) - D_{t=0}^{-(1-3/2)}y(0) = 0(1.23)$$

we will assume that constants $D_{t=0}^{-(2-3/2)}y(0)\ and\ D_{t=0}^{-(1-3/2)}y(0)$ exist and call them $c_1\ and\ c_2$, respectively. Then (1.23) becomes

$$s^{3/2}Y(s) - c_1 s - c_2 = 0.$$

Solving for $Y(s)$ we obtain:

$$Y(s) = \frac{c_1 s}{s^{3/2}} + \frac{c_2}{s^{3/2}}$$

Finally, using Table 1.1 we find the inverse Laplace of $Y(s)$ and conclude that

$$y(t) = \mathcal{L}^{-1}\left\{\frac{c_1 s}{s^{3/2}}\right\} + \mathcal{L}^{-1}\left\{\frac{c_2}{s^{3/2}}\right\}$$

$$= \frac{c_1}{\Gamma\left(\frac{1}{2}\right)} t^{-1/2} + \frac{c_2}{\Gamma\left(\frac{3}{2}\right)} t^{1/2}.$$

Before starting the second method we clarify some concepts:

B. Convolution of fractional green's functions

Nowwe will discuss some interesting results involving fractional green's functions. The first theoremprovingthat the convolution of two fractional green's functions is also a fractional green's functions.

Theorem 1[3]: let x(t) be piecewise continuous on J' and integrable and exponential order on J .let

$$\left[D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0\right]y(t) = x(t).$$

$$D^j y(0) = 0, \qquad j = 0,1,\ldots,N-1. \qquad (1.24)$$

be a fractional differential system of order (n,q),where N is the smallest integer greater than nv. let $p(x) = x^n + a_1 x^{n-1} + \cdots + a_n$. Be the indicial polynomial and let $k(t) = \mathcal{L}^{-1}\{p^{-1}(s^v)\}$ be the fractional green's function then

$$y(t) = \int_0^t k(t-x) f(x) dx.$$

Theorem 2[3]: let

$$P(D^v) = D^{nv} + a_1 D^{(n-1)v} + \cdots + a_n D^0 \qquad (1.25)$$

Be a fractional differential operator of order (n,q) with fractional green's function $K_P(t)$, and let

$$Q(D^v) = D^{mv} + b_1 D^{(m-1)v} + \cdots + b_m D^0 \qquad (1.26)$$

be a fractional differential operator of order (m,q) with fractional green's function $K_Q(t)$, let

$$R(x) = Q(x)P(x) \qquad (1.27)$$

and let

$$R(D^v) = D^{(m+n)v} + c_1 D^{(m+n-1)v} + \cdots + c_{n+m} D^0, \ (1.28)$$

a fractional differential operator of order (m+n,q) then If $K_Q(t)$ is a fractional green's function associated with $R(D^v)$,

$$K_R(t) = \int_0^t K_Q(t-x)K_P(x)dx. \qquad (1.29)$$

Proof:[3] We know that

$$\mathcal{L}\{K_Q(t)\}\mathcal{L}\{K_p(t)\} = \frac{1}{Q(s^v)}\frac{1}{P(s^v)}$$

and

$$R(s^v) = Q(s^v)P(s^v).$$

But is $R^{-1}(s^v)$ a Laplace transform of $K_R(t).thus$

$$\mathcal{L}\{K_Q(t)\}\mathcal{L}\{K_p(t)\} = \mathcal{L}\{K_R(t)\},$$

and by the convolution theorem of the Laplace transform,

$$K_R(t) = \int_0^t K_Q(t-x)K_P(x)dx. \qquad (1.30)$$

Corollary 1[3]. If $P(D^v), Q(D^v), and\ R(D^v)$ are the fractional operators of (1.25), (1.26), and (1.28) and $K_P, K_Q\ and\ K_R$ are their respective fractional green's function, then

$$Q(D^v)K_R(t) = K_P(t)(1.31a)$$

and

$$P(D^v)K_R(t) = K_Q(t).(1.31b)$$

If P(x) is a polynomial of degree n≥ 1,and if q is any positive integer, there exists a polynomial Q of degree n(q-1) such that

$$Q(x)P(x)$$

Is a polynomial of degree n in x^q.

We have

$$R(D^v) = Q(D^v)P(D^v),$$

and

$$R(D^v) = T(D) = D^n + d_1D^{n-1} + \cdots + d_nD^0.$$

That is, T(D) is an ordinary differential operator .

Let H(t) be the one-sided green's function associated with T.

Then

$$H(t) = \int_0^t K_Q(t-x)K_P(x)dx. \qquad (1.32)$$

Corollary 2[3]. if $P(D^v)$ is a fractional differential operator of order (n,q), there exists a fractional differential operator $Q(D^v)$ of order $(n(q-1),q)$ such that the convolution of their fractional green's functions is a one- sided green's function of an ordinary differential operator of order n.In particular, corollary 1 implies that :

$$Q(D^v)H(t) = K_P(t) \qquad (1.33a)$$

and

$$P(D^v)H(t) = K_Q(t) \qquad (1.33b)$$

C. The techniques of transformation of second method [3]:

We explain the method to transformation from fractional differential equations to ordinary differential equation as following:

Suppose that we wish to solve the fractional differential system of order (n,q)

$$[D^{nv} + a_1D^{(n-1)v} + \cdots + a_nD^0]y(t) = x(t)(1.34a)$$

$$y(0) = D_y(0) = \cdots = D^{N-1}y(0) = 0, \qquad (1.34b)$$

Where N is the smallest integer with the property that N $\geq nv$, and x(t) is piecewise continuous on J', and of exponential order on J. let

$$P(x) = x^n + a_1x^{n-1} + \cdots + a_n \quad (1.35)$$

Be the indicial polynomial. [Then we may write (1.34a) as

$$P(D^v)\, y(t) = x(t).]$$

Given a polynomial P of degree n in x, we may construct two polynomials, T and Q, such that

$$T(x^q) = Q(x)P(x), \quad (1.36)$$

Where Q is a polynomial of degree n(q-1) in x, and T is a polynomial of degree n in x^q. Choose Pas the P(x) of (2.2). For the ordinary differential operator:

$$T(D) = D^n + d_1D^{n-1} + \cdots + d_nD^0(1.37)$$

We may construct its one-sided green's function, say H(t). Then form (1.33a) we see that

$$Q(D^v)H(t) = K_P(t)(1.38)$$

Thus we have obtained the fractional green's function K_P of $P(D^v)$ by applying the fractional operator to the known functionH(t). Then the solution of (1.34) is given by

$$y(t) = \int_0^t K_Q(t-x)K_P(x)dx. \quad (1.39)$$

So we see that the only place where we needed the fractional calculus was when we had to compute fractional derivatives of a known function.

Example3: consider the fractional differential system of order (2,3),

$$[D^{2v} - D^v + D^0]y(t) = f(t) \quad (1.40a)$$

$$y(0) = 0. \quad (1.40b)$$

(Here N=1.) Then

$$P(x) = x^2 - 4x + 4 \quad (1.41)$$

Is the indicial polynomial associated with (1.40a). Using (1.41) as the polynomial "P(x)" we have

$$Q(x) = x^4 + 4x^3 + 12x^2 + 16x + 16$$

and

$$R(x) = Q(x)P(x) = x^6 - 16x^3 + 64$$

$$T(x) = R(x^v) = Q(x^v)P(x^v)$$

$$= x^2 - 16x + 64.$$

The one-sided green's function H(t) associated with the ordinary differential operator

$$T(D) = D^2 - 16D + 64D^0.$$

Now recall that

$$D^v(te^{kt}) = tE_t(-v,k) + vE_t(1-v,k)$$

Hence

$$Q(D^v)H(t) = [D^{4v} + 4D^{3v} + 12D^{2v} + 16D^v + 16](te^{8t})$$

$$= t[E_t(-4v,8) + 6E_t(-3v,8) + 12E_t(-2v,8) + 16E_t(-v,8) + 4v[E_t(-v,8) + E_t(0,8) + 6E_t(v,8) + 4E_t(2v,8)].$$

We see from (1.38) that the expression above is $K_P(t)$. Therefore,

$$y(t) = \int_0^t \{(t-x)[E_{t-x}(-4v,8) + 6E_{t-x}(-3v,8) + 12E_{t-x}(-2v,8) + 16E_{t-x}(-v,8)] + 4v[E_{t-x}(-v,8) + 3E_{t-x}(0,8) + 6E_{t-x}(v,8) + 4E_{t-x}(2v,8)]\}f(x)dx.$$

(1.42)

is the solution given by (1.39)

III. CONCLUSION

In this paper, W prove that how the solution of the fractional differential equation system can be converted into a problem in ordinary differential equation in two method.

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