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Generalize Prey-Predator Model with Reserve and Unreserved Area

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ABSTRACT

In this paper we generalized a prey-predator model with prey dispersal in a two patch environment, one is assumed to be free zone and other is reserved zone where fishing and other extractive activities are prohibited. The steady states of the system are determined. The local and global stability analysis has been carried out.
Keywords: Stability, Equilibrium, Prey-Predator Model, Liapunov Function

I. INTRODUCTION

Biological resources are renewable resources. Bio economic modeling of the exploitation of the biological resources such as fisheries and floristries has gained importance in recent years. As we know the biological species have been driven to extinct due to either natural or manmade reasons such as over exploitation, over predation, unregulated harvesting, environment pollution etc. One potential solution of this problem in the creation of marine resources where fishing and other extractive activities are prohibited. Mathematical model of ecological system reflecting these problems has been carried out Kar and Swarnakamal [10].Nonlinear behavior of predator-prey model with refuge protecting a constant proportion of prey and wit temperature dependent parameters chosen appropriately for amite interaction on fruit species discussed by Colling [7]. Effect of two interacting population on resource following generalized logistic growth proposed by Singh et.al. [3]. Biological resources are renewable resources. Economic and biological aspect of renewable resources management has been considered by Clark [4]. A mathematical model of selective harvesting in a prey-predator fishery with time delay given by Kar [9]. A prey-predator model with a reserved area given in Dubey et.al.[1]. Persistence and extinction of one prey, two predator system analyzed and proposed by Dubey et.al.[2]. Optimal harvesting policy of prey-predator model considered by Zhang et.al.[11]. Prey –predator model with a generalized transmission function analyzed by Mehta et.al. [6]. In this paper we generalized the model of [1] when predator is wholly depends on the prey species. The model takes the form
Model equation is

$$\frac{dx}{dt} = rx\left(1-\left(\frac{x}{K}\right)^n\right) - \sigma_1 x - \sigma_2 y - \beta_1 xz$$

$$\frac{dy}{dt} = sy\left(1-\frac{y}{L}\right) + \sigma_1 x - \sigma_2 y$$

$$\frac{dz}{dt} = Q(Z) - \beta_0 x$$

$$x(0) \geq 0, y(0) \geq 0, z(0) \geq 0 \qquad (1)$$

Where
$\boldsymbol{x(t)}$ be the density of prey species in unreserved zone,
$\boldsymbol{y(t)}$ be the density of prey species in reserved zone and
$\boldsymbol{z(t)}$ the density of the predator species at any time $t \geq \mathbf{0}$, σ_1 be the migration rate coefficient of the prey species from unreserved to reserved zone,

$\boldsymbol{\sigma_2}$ the migration rate coefficient of the prey species from reserved to unreserved zone. Prey species in reserved and unreserved zone is assumed to be logistically.

r and s are intrinsic growth rate coefficient of prey species in unreserved and reserved zones respectively;
K and L are their respective carrying capacities.

$\boldsymbol{\beta_1}$ is the depletion rate coefficient of the prey species due to the predator , and β_0 is the natural death rate coefficient of the predator species.

In model (1), the function $Q(Z)$ represents the growth rate of predator. The model (1) is analyzed when predator is wholly dependent on the prey species, so that $Q(Z) = \beta_2 xz$ then eqution (3) of model (1) is

$$\frac{dz}{dt} = \beta_2 xz - \beta_0 x$$

Now above model becomes

$$\frac{dx}{dt} = rx\left(1 - \left(\frac{x}{K}\right)^n\right) - \sigma_1 x - \sigma_2 y - \beta_1 xz$$

$$\frac{dy}{dt} = sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y$$

$$\frac{dz}{dt} = \beta_2 xz - \beta_0 x \qquad (1a)$$

II. METHODS AND MATERIAL

Existence of Equilibria

There are three nonnegative equilibria namely $E_0(0,0,0)$, $E_1(\hat{x}, \hat{y}, 0)$ and $\bar{E}(\bar{x}, \bar{y}, \bar{z})$.

The equilibrium E_0 exist obviously. We prove the existence of E_1 and $\bar{E}$ as follows:

Existence of $\boldsymbol{E_1}(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \boldsymbol{0})$

Here $\hat{x}$ and $\hat{y}$ are the positive solution of the following algebraic equations:

$$rx\left(1 - \left(\frac{x}{K}\right)^n\right) - \sigma_1 x + \sigma_2 y - \beta_1 xz = 0 \qquad (2)$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0 \qquad (3)$$

From (2)

$$y = \frac{1}{\sigma_2}\left(-(r - \sigma_1 x) + \frac{rx^{n-1}}{K^n}\right) \qquad (4)$$

Substituting the value of y from (4) into equation (3) we get

$$Ax^{2n+1} + Bx^{n+1} + cx^n + Dx + E = 0 \qquad (5)$$

Where

$$A = \frac{sr^2}{L\sigma_2^2 K^{2n}} \quad , \quad B = \frac{-(r - \sigma_1)}{K\sigma_2^2 L} 2sr \, ,$$

$$C = \frac{r}{K^n}\left(\frac{-s}{\sigma_2} + \frac{r}{K}\right),$$

$$D = (r - \sigma_1)^2 \frac{s}{\sigma_2^2} \quad \text{and} \quad E = \frac{(r-\sigma_1)(s-\sigma_2)}{\sigma_2} - \sigma_1$$

It is clear that (5) has a unique solution $x = x^*$ if the following inequality hold:
$(r - \sigma_1)(s - \sigma_2) < \sigma_1 \sigma_2$.

From the model system (1a) we note that if there is no migration of species from reserved to unreserved zone (i.e. $\sigma_2 = 0$) and $(r - \sigma_1) < 0, then\ \frac{dx}{dt} < 0$. similarly if there is no migration from of the prey species from unreserved to reserved zone

(i.e. $\sigma_1 = 0$) and $(s - \sigma_2) < 0$, then $\frac{dy}{dt} < 0$. Hence it is natural to assume that

$$r > \sigma_1 \quad \text{and} \quad s > \sigma_2.$$

Knowing the value of $\hat{x}$, the value of $\hat{y}$ ca n be computed from equation (4). It may also be noted for $\hat{y}$ to be positive we must have

$$\hat{x}^n > \frac{K^n}{r}(r - \sigma_1)$$

Existence of $\bar{E}(\bar{x}, \bar{y}, \bar{z})$

Here $\bar{x}, \bar{y}$ and $\bar{z}$ are the positive solution of the following algebraic equations:

$$rx\left(1 - \left(\frac{x}{K}\right)^n\right) - \sigma_1 x - \sigma_2 y - \beta_1 xz = 0 \qquad (6a)$$

$$sy\left(1 - \frac{y}{L}\right) + \sigma_1 x - \sigma_2 y = 0 \qquad (6b)$$

$$\beta_2 xz - \beta_0 x = 0 \qquad (6c)$$

Solving the above equation, we get ,

$$\bar{x} = \frac{\beta_0}{\beta_2} \qquad (7a)$$

$$\bar{y} = \frac{L}{2s}\left[(s - \sigma_2) \pm \sqrt{(s - \sigma_2)^2 + \frac{4s\sigma_1\beta_0}{2\beta_2}}\right] \qquad (7b)$$

$$\bar{z} = \frac{\beta_2}{\beta_0\beta_1}\left[\sigma_2\bar{y} + (r-\sigma_1)\frac{\beta_0}{\beta_2} - \frac{r\beta_0^{n+1}}{K^n\beta_2^{n+1}}\right] \qquad (7c)$$

For $\bar{z}$ to be positive , we must have

$$\sigma_2\bar{y} + (r-\sigma_1)\frac{\beta_0}{\beta_2} > \frac{r\beta_0^{n+1}}{K^n\beta_2^{n+1}} \qquad (8)$$

Equation (8) gives a threshold value of the carrying capacity of the free access zone for the survival of predators.

In the following lemma, we show that all solutions of the model (2) are nonnegative and bounded.

$$\Omega = \left\{(x,y,z) \in CR_3^+ : 0 < x+y+z \le \frac{\mu}{\eta}\right\}$$

Is a region of the attraction for all solutions initiating of the positive orthant, where η is a constant such that

$$0 < \eta < \beta_0,\ \ \mu = \frac{K^n}{4r}(r+\eta)^2 + \frac{L}{4s}(s+\eta)^2,$$
$$\beta_1 \ge \beta_2$$

Proof: Let $\omega(t) = x(t) + y(t) + z(t)$ and $\eta > 0$ be a constant. Then

$$\frac{d\omega}{dt} + \eta\omega = x(r+\eta) + y(s+\eta) - (\beta_0-\eta)z - \frac{rx^{n+1}}{K^n} - \frac{sy^2}{L} - xz(\beta_1-\beta_2) \quad (9)$$

Since β_1 is the depletion rate coefficient of prey due to its intake by the predator and β_2 is the growth rate coefficient of predator to its interaction with their prey, and hence it is natural to assume that $\beta_1 \ge \beta_2$. Now choose η such that $0 < \eta < \beta_0$, then equation (9) can be written as

$$\frac{d\omega}{dt} + \eta\omega \le x(r+\eta) + y(s+\eta) - \frac{rx^{n+1}}{K^n} - \frac{sy^2}{L}$$

$$\begin{aligned}\frac{d\omega}{dt} + \eta\omega = \frac{K^n}{4r}(r+\eta)^2 &- \frac{r}{K^n}\left[x^{(n+1)/2} - \frac{K^n}{2r}(r+\eta)^{(1-n)/2}\right]^2 \\ &- \frac{K^n}{4r}(r+\eta)^2[1-x^{1-n}] \\ &+ \frac{L}{4s}(s+\eta)^2 - \frac{s}{L}\left[y - \frac{L}{2s}(s+\eta)\right]^2\end{aligned}$$

$$\frac{d\omega}{dt} + \eta\omega \le \frac{K^n}{4r}(r+\eta)^2 + \frac{L}{4s}(s+\eta)^2 = \mu(say)$$

By using the differential inequality [5] , we obtain
$0 < \omega\big(x(t),y(t),z(t)\big) \le$
$\frac{\mu}{\eta}(1-e^{-\eta t}) + \big(x(0),y(0),z(0)\big)e^{-\eta t}.$

Taking limit when $t \to \infty$, we have , $0 < \omega(t) \le \frac{\mu}{\eta}$, proving the lemma.

III. RESULTS AND DISCUSSION

Stability Analysis : Vibrational matrix of model (2) is

$$J = \begin{bmatrix} r - \left(\frac{n+1}{K}\right)rK^n - \sigma_1 - \beta_1 z & \sigma_2 & -\beta_1 x \\ \sigma_1 & s - \frac{2sy}{L} - \sigma_2 & 0 \\ \beta_2 z & 0 & \beta_2 x - \beta_0 \end{bmatrix}$$

At $E_0(0,0,0)$

$$J_o = \begin{bmatrix} r-\sigma_1 & \sigma_2 & 0 \\ \sigma_1 & s-\sigma_2 & 0 \\ 0 & 0 & -\beta_0 \end{bmatrix}$$

By the characteristic equation of matrix E_0 is a saddle point with stable manifold locally in the z-direction.

At $E_1(\hat{x},\hat{y},0)$

$$J_1 = \begin{bmatrix} r - \left(\frac{n+1}{K}\right)rK^n - \sigma_1 & \sigma_2 & -\beta_1 x \\ \sigma_1 & s - \frac{2sy}{L} - \sigma_2 & 0 \\ 0 & 0 & \beta_2 x - \beta_0 \end{bmatrix}$$

By the characteristic equation of matrix

(a) If $\beta_2\hat{x} > \beta_0$ then E_1 is a saddle is a saddle point with stable manifold locally in the xy plane and with unstable manifold locally in the z-direction.
(b) $\beta_2\hat{x} < \beta_0$ then E_1 is locally asymptotically stable.

In the following theorem, we show that the model (2) cannot have any closed trajectory in the interior of the positive quadrant of the xy- plane

Theorem 1: The model system (2) cannot have any periodic solution in the interior of the positive quadrant of the xy-plane.

Proof: Let $H(x,y)=\frac{1}{xy}$. Clearly $H(x,y)$ is a positive in the interior of the positive quadrant of the xy- plane.
Let

$$h_1(x,y)=rx\left(1-\left(\frac{x}{K}\right)^n\right)-\sigma_1 x+\sigma_2 y$$

$$h_2(x,y)=sy\left(1-\frac{y}{L}\right)+\sigma_1 x-\sigma_2 y$$

Then

$$\Delta(x,y)=\frac{\partial}{\partial x}(h_1 H)+\frac{\partial}{\partial y}(h_2 H)$$

$$\Delta(x,y)=\frac{-n}{y}\left[\frac{rx^{n-1}}{K^n}+\frac{\sigma_2 y}{x^2}\right]-\frac{1}{x}\left[\frac{s}{L}+\frac{\sigma_1 x}{y^2}\right]<0$$

From the above equation, we note that $\Delta(x,y)$ does not change the sign and is not identically zero in the interior of the positive quadrant of the xy- plane. By Dulac Bindixon criteria , it follows that there is no closed trajectory in the interior of the positive quadrant of the xy- plane , hence the theorem follows.

In following theorem , we are able to show that $\bar{E}$ is globally asymptotically stable.

Theorem 2: The interior equilibrium of $\overline{E}$ is globally asymptotically stable with respect to all solution initiating in the interior of the positive orthant.

Proof: Consider the following positive definite function about $\overline{E}$.

$$W(t)=\left(x-\bar{x}-In\frac{x}{\bar{x}}\right)+c_1\left(y-\bar{y}-In\frac{y}{\bar{y}}\right)+c_2\left(z-\bar{z}-In\frac{z}{\bar{z}}\right)$$

Differentiating W with respect to time t along the solution of model (2) , we get

$$W(t)=\frac{-r}{K^n}(x-\bar{x})(x^n-\bar{x}^n)+ +\sigma_2(x-\bar{x})(\frac{\bar{x}y-\bar{y}x}{x\bar{x}})-(c_2\beta_2-\beta_1)(z-\bar{z})(x-\bar{x}) -\frac{s}{L}(y-\bar{y})^2+c_1\sigma_1\,(y-\bar{y})\;\;(\frac{\bar{y}x-\bar{x}y}{y\bar{y}})$$

Choose $c_1=\frac{\sigma_2\bar{y}}{\sigma_1\bar{x}}$ and $c_2=\frac{\beta_1}{\beta_2}$, then

$$\frac{dW}{dt}=\frac{-r}{K^n}(x-\bar{x})(x^n-\bar{x}^n)-\frac{\bar{y}\sigma_2 s}{L}(y-\bar{y})^2 -\frac{\sigma_2}{x\bar{x}y}(\bar{x}y-\bar{y}x)^2<0$$

Which is negative definite. Hence W is a Liapunov function [8] with respect to $\bar{E}$ whose domain contains the region of attraction Ω , proving the theorem.
If we put n=1 we get the result of B. Dubey [1].

IV. CONCLUSION

In this paper we generalized the prey-predator model, in the case of when predator is wholly depends on the prey species. Criteria for coexistence of prey –predator are obtained using the stability theory.

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