© 2015 IJSRSET | Volume 1 | Issue 5 | Print ISSN : 2395-1990 | Online ISSN : 2394-4099
Themed Section: Science and Technology

'Useful' Renyi's Information Measure of Order α, Type β And Source Coding Theorem

[1]Dhanesh Garg, [2]Satish Kumar
Maharishi Markendeshwar University, Mullana-Ambala, Haryana, India

ABSTRACT

A parametric mean length is defined as the quantity

$$ {}_{\alpha\beta}L_u = \frac{1}{1-\alpha}\log_D\left[\frac{\sum (u_i p_i)^{\beta} D^{-n_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}}\right], $$

where $\alpha > 0\ (\neq 1)$, $\beta > 0, u_i > 0, D > 1$ is an integer, $\sum p_i = 1$.This being the useful mean length of code words weighted by utilities, u_i. Lower and Upper bounds for ${}_{\alpha\beta}L_u$ are derived in terms of 'useful' Renyi's entropy of order α type β.

AMS Subject classification: 94A15, 94A17, 94A24, 26D15.

Keywords: Tsalli's Entropy, Useful Tsalli's entropy, Utilities, Kraft inequality, Holder's inequality.

I. INTRODUCTION

Consider the following model for a random experiment S,

$$ S_N = [E;\ P;\ U] $$

where $E = (E_1, E_2, ..., E_N)$ is a finite system of events happening with respective probabilities $P = (p_1, p_2, ..., p_N)$, $p_i \geq 0$, $\sum p_i = 1$ and credited with utilities $U = (u_1, u_2, ..., u_N)$, $u_i > 0$, $i = 1, 2, \ldots, N$. Denote the model by S_N, where,

$$ S_N = \begin{bmatrix} E_1, E_2, \ldots, E_N \\ p_1, p_2, \ldots, p_N \\ u_1, u_2, \ldots, u_N \end{bmatrix}. \tag{1.1} $$

We call (1.1) a Utility Information Scheme (UIS). Belis and Guiasu [2] proposed a measure of information called 'useful information' for this scheme, given by

$$ H(U;\ P) = -\sum u_i p_i \log p_i , \tag{1.2} $$

where $H(U;\ P)$ reduces to Shannon's [8] entropy when the utility aspect of the scheme is ignored i.e., when $u_i = 1$ for each *i*. Throughout the paper, $\sum$ will stand for $\sum_{i=1}^{N}$ unless otherwise stated and logarithms are taken to base $D\ (D > 1)$.

Guiasu and Picard [4] considered the problem of encoding the outcomes in (1.1) by means of a prefix code with codewords $w_1, w_2, \ldots, w_N$ having lengths $n_1, n_2, \ldots, n_N$ and satisfying Kraft's inequality [9] .

$$ \sum_{i=1}^{N} D^{-n_i} \leq 1. \tag{1.3} $$

Where D is the size of the code alphabet. The useful mean length L_u of code was defined as :

$$ L_u = \frac{\sum u_i n_i p_i}{\sum u_i p_i} , \tag{1.4} $$

and the authors obtained bounds for it in terms of $H(U;\, P)$. Generalized coding theorems by considering different generalized measures under condition (1.3) of unique decipherability were investigated by several authors, see for instance the papers [1,3,15].

In this paper, we study some coding theorems by considering a new function depending on the parameters α, β and a utility function. Our motivation for studying this new function is that it generalizes 'useful' information measure already existing in the literature such Renyi's entropy.

II. CODING THEOREMS

In this section, we define a new information measure as :

$$_{\alpha\beta}H(U;\, P)=\frac{1}{1-\alpha}\log_D\left[\frac{\sum\left(u_i p_i^{\alpha}\right)^{\beta}}{\sum\left(u_i p_i\right)^{\beta}}\right], \qquad (2.1)$$

where

$\beta>0, \alpha>0\ (\neq 1),\ u_i>0,\ p_i\geq 0,\ i=1, 2, ..., N$ and $\sum p_i=1$.

(i) If $\beta=1$, Then (2.1) becomes a 'useful' information measure

i.e., $$_{\alpha}H(U;\, P)=\frac{1}{1-\alpha}\log_D\left[\frac{\sum u_i p_i^{\alpha}}{\sum u_i p_i}\right] \qquad (2.2)$$

which is studied Hooda[1].

(ii) When $u_i=1$ for each *i*, i.e., when the utility aspect is ignored, $\sum p_i=1$, and $\beta=1$, then (2.1) reduces to Renyi's entropy .

i.e., $$_{\alpha}H(P)=\frac{1}{1-\alpha}\log_D\left[\sum p_i^{\alpha}\right]. \qquad (2.3)$$

(iii) When $\alpha\to 1$, and $\beta=1$, then (2.1) reduces to a measure of 'useful' information due to Hooda and Bhaker [1].

i.e., $$H(U;P)=-\frac{\sum u_i p_i \log p_i}{\sum u_i p_i}. \qquad (2.4)$$

(iv) When $u_i=1$ for each *i*, then (2.1) reduced to Satish and Arun [6] entropy.

i.e., $$_{\alpha\beta}H(U;\, P)=\frac{1}{1-\alpha}\log_D\left[\frac{\sum p_i^{\alpha\beta}}{\sum p_i^{\beta}}\right]. \qquad (2.5)$$

(v) When $u_i=1$ for each *i*, i.e., When the utility aspect is ignored, $\sum p_i=1$, $\beta=1$, and $\alpha\to 1$, the measure (2.1) reduces to Shannon's entropy [8].

i.e., $$H(P)=-\sum p_i \log p_i. \qquad (2.6)$$

Further consider,

Definition: The 'useful' mean length $_{\alpha\beta}L_u$ with respect to 'useful' R-norm information measure is defined as :

$$_{\alpha\beta}L_u=\frac{1}{1-\alpha}\log_D\left[\frac{\sum\left(u_i p_i\right)^{\beta}D^{-n_i(\alpha-1)}}{\sum\left(u_i p_i\right)^{\beta}}\right], \qquad (2.7)$$

under the condition, $$\sum\left(u_i D^{-n_i\alpha}\right)^{\beta}\leq\sum\left(u_i p_i^{\alpha}\right)^{\beta}. \qquad (2.8)$$

Clearly the inequality (2.8) is the generalization of Kraft's inequality (1.3). A code satisfying (2.8) would be termed as a useful personal probability code D (D>2) is the size of the code alphabet. When, $u_i=1$ for each i and $\beta=1, \alpha=1$, (2.8) reduces to (1.3).

(i) For $u_i=1$ for each i and $\beta=1$, and $\alpha\to 1$, $_{\alpha}L_u$ becomes the optimal code length defined by Shannon [8].

(ii) For $u_i=1$ for each i and $\beta=1$, then (2.7) becomes a new mean code word length corresponding to the Tsalli's entropy.

i.e., $$_{\alpha}L=\frac{1}{1-\alpha}\log_D\left[\sum p_i D^{-n_i(\alpha-1)}\right]. \qquad (2.9)$$

(iii) If $\beta=1$, then (2.7) becomes a new mean codewords length corresponding to the entropy (2.2).

i.e., $$_{\alpha}L_u=\frac{1}{1-\alpha}\log_D\left[\frac{\sum u_i p_i D^{-n_i(\alpha-1)}}{\sum u_i p_i}\right].$$

(iv) If $u_i=1$, then (2.7) becomes a mean codewords length corresponding to the entropy (2.5).

i.e., $$_{\alpha\beta}L=\frac{1}{1-\alpha}\log_D\left[\frac{\sum p_i^{\beta}D^{-n_i(\alpha-1)}}{\sum p_i^{\beta}}\right].$$

We establish a result, that in a sense, provides a characterization of ${}_{\alpha\beta}H(U;\, P)$ under the condition of unique decipherability.

Theorem 2.1. Let $u_i, p_i, n_i,\ i=1,2,...,N$, satisfy the inequality (2.8). Then

$${}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H(U;P), \quad 1\neq\alpha>0, \beta>0. \qquad (2.10)$$

Proof: By Holder's inequality, we have

$$\left(\sum_{i=1}^{N} x_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{N} y_i^q\right)^{\frac{1}{q}} \leq \sum_{i=1}^{N} x_i y_i\,, \qquad (2.11)$$

where $p^{-1}+q^{-1}=1;\ p(\neq 0)<1, q<0$ or $q(\neq 0)<1, p<0; x_i, y_i>0$ for each i.

Setting, $p=\frac{(\alpha-1)}{\alpha}, q=1-\alpha$, and

$$x_i=\left(\frac{(u_i p_i)^\beta}{\sum(u_i p_i)^\beta}\right)^{\frac{\alpha}{\alpha-1}} D^{-n_i\alpha}, \quad y_i=\left(\frac{(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}\right)^{\frac{1}{\alpha-1}},$$
(2.12)

Putting these values in (2.11) and using the inequality (2.8), we get

$$\left(\frac{\sum(u_i p_i)^\beta\ D^{-n_i(\alpha-1)}}{\sum(u_i p_i)^\beta}\right)^{\frac{\alpha}{\alpha-1}} \left(\frac{\sum(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}\right)^{\frac{1}{\alpha-1}} \leq \frac{\sum(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}$$
(2.13)

It implies

$$\left(\frac{\sum(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}\right)^{\frac{\alpha}{1-\alpha}} \leq \left(\frac{\sum(u_i p_i)^\beta\ D^{-n_i(\alpha-1)}}{\sum(u_i p_i)^\beta}\right)^{\frac{\alpha}{\alpha-1}} \qquad (2.14)$$

Taking $\log_D$ both sides, we get

$$\frac{1}{1-\alpha}\log_D\left[\frac{\sum(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}\right] \leq \frac{1}{1-\alpha}\log_D\left[\frac{\sum(u_i p_i)^\beta\ D^{-n_i(\alpha-1)}}{\sum(u_i p_i)^\beta}\right]$$
(2.15)

It is clear that the equality in (2.10) is true if and only if

$$D^{-n_i}=p_i^\beta$$

which implies that

$$n_i=\log_D\frac{1}{p_i^\beta} \qquad (2.16)$$

Thus, it is always possible to have a codeword satisfying the requirement

$$\log_D\frac{1}{p_i^\beta} \leq n_i < \log_D\frac{1}{p_i^\beta}+1,$$

which is equivalent to

$$\frac{1}{p_i^\beta} \leq D^{n_i} < \frac{D}{p_i^\beta}\ . \qquad (2.17)$$

In the following theorem, we give an upper bound for ${}_{\alpha\beta}L_u$ in terms of ${}_{\alpha\beta}H(U;P)$.

Theorem 2.2. By properly choosing the lengths $n_1, n_2, ..., n_N$ in the code of Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisy the following inequality:

$${}_{\alpha\beta}L_u < D^{(1-\alpha)}\ {}_{\alpha\beta}H(U;P)+\frac{1}{\alpha-1}\left(1-D^{(1-\alpha)}\right)$$
(2.18)

Proof: From (2.17), it is clear that

$$D^{-n_i} > D^{-1}\,p_i^\beta\ . \qquad (2.19)$$

We have again the following two possibilities.

(i) Let $\alpha>1$. Raising both sides of (2.19) to the power $(\alpha-1)$, we have

$$D^{-n_i(\alpha-1)} > D^{1-\alpha}\,p_i^{\beta(\alpha-1)}\ .$$

Multiplying both sides by $(u_i p_i)^\beta$ and then summing over i. we get

$$\sum(u_i p_i)^\beta D^{-n_i(\alpha-1)} > D^{(1-\alpha)}\sum\left(u_i p_i^\alpha\right)^\beta\ . \qquad (2.20)$$

Obviously (2.20) can be written as

$$\frac{\sum(u_i p_i)^\beta\ D^{-n_i(\alpha-1)}}{\sum(u_i p_i)^\beta} > D^{(1-\alpha)}\ \frac{\sum(u_i p_i^\alpha)^\beta}{\sum(u_i p_i)^\beta}. \qquad (2.21)$$

Since $\alpha-1>0$ for $\alpha>1$, we get the inequality (2.18) from (2.21).

(ii) If $0<\alpha<1$, the proof follows similarly. But the inequality (2.21) is reversed.

Theorem 2.3. For arbitrary $N\in\mathbb{N}$, $1\neq\alpha>0, \beta>0$, and for every codeword lengths $n_i, i=1,2,...,N$ of

Theorem 2.1, ${}_{\alpha\beta}L_u$ can be made to satisy the following inequality:

$$ {}_{\alpha\beta}L_u \geq {}_{\alpha\beta}H(U;P) > {}_{\alpha\beta}H(U;P) + \frac{1}{1-\alpha} \cdot \quad (2.22)$$

Proof: Suppose,

$$\bar{n}_i = \log_D \frac{1}{p_i^{\beta}}, \ \beta > 0. \quad (2.23)$$

Clearly $\bar{n}_i$ and $\bar{n}_i + 1$ satisfy the equality in Holder's inequality (2.11). Moreover, $\bar{n}_i$ satisfies (2.8). Suppose $\bar{n}_i$ is the unique integer between $\bar{n}_i$ and $\bar{n}_i + 1$, then obviously, $\bar{n}_i$ satisfies (2.8).

Since $1 \neq \alpha > 0$, $\beta > 0$, we have

$$\frac{\sum (u_i p_i)^{\beta} D^{-n_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}} \leq \frac{\sum (u_i p_i)^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}}$$

$$< D\left(\frac{\sum (u_i p_i)^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}}\right) \quad (2.24)$$

Since, $\dfrac{\sum (u_i p_i)^{\beta} D^{-\bar{n}_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}} = \dfrac{\sum (u_i p_i^{\alpha})^{\beta}}{\sum (u_i p_i)^{\beta}}$.

Hence (2.24) becomes

$$\frac{\sum (u_i p_i)^{\beta} D^{-n_i(\alpha-1)}}{\sum (u_i p_i)^{\beta}} \leq \left(\frac{\sum (u_i p_i^{\alpha})^{\beta}}{\sum (u_i p_i)^{\beta}}\right) < D\left(\frac{\sum (u_i p_i^{\alpha})^{\beta}}{\sum (u_i p_i)^{\beta}}\right).$$

Which gives (2.22).

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