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Study of Numerical Solution of Fourth Order Ordinary Differential Equations by fifth order Runge-Kutta Method

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ABSTRACT
In this paper we present fifth order Runge-Kutta method (RK5) for solving initial value problems of fourth order ordinary differential equations. In this study RK5 method is quite efficient and practically well suited for solving boundary value problems. All mathematical calculation performed by MATLAB software for better accuracy and result. The result obtained, from numerical examples, shows that this method more efficient and accurate. These methods are preferable to some existing methods because of their simplicity, accuracy and less computational cost involved.
Keywords : Runge-Kutta method, MATLAB, IVP, BVP

I. INTRODUCTION

We face many problems in science and engineering in terms of differential equations. A differential equation is an equation involved in a relation between an unknown function and one or more its derivatives[4,11]. Equations involving its derivative of only one independent variable are called ordinary differential equations and may be classified as either initial value problems (IVP) or boundary value problems (BVP). Fourth order boundary value problems occur in a number of areas of applied mathematics among which are fluid mechanics, elasticity and quantum mechanics as well as science and engineering[1-3]. Only small class of differential equations can be solved by analytic methods. Hence, several authors have investigated some numerical techniques for solving numerical value problems. A more robust and intricate numerical technique is the Runge-Kutta method. This method is most widely used one science it gives reliable starting values and is particularly suitable when the computation of higher derivatives is complicated[6,7,8].

The general forth order differential equations (ODEs) of the form

$$y^{(4)} = f(x, y, y', y'', y'''), \qquad 0 \le x \le L \quad ...(1)$$

are considered in the paper. Eq. (1) can be solved by reducing it to its equivalent first order system as mentioned. However, this approach suffers some setbacks such as evaluation of too many functions and heavy computation [9,10].

In this paper we also discussed accuracy analysis of numerical solutions of initial value problems for ordinary differential equations. (M.D. Islam 2015) discussed accurate solutions of initial value problems for ordinary differential equations with fourth order Runge-Kutta method [6,7].

(Ogunrinde et al. 2012) studied on some numerical methods for solving initial value problems in ordinary differential equations. Gemechis File and Tesfaye Aga 2016 considered fourth order Runge-Kutta method for solving Riccati differential equations. In this paper, we introduce fifth order RK method for solving fourth order ordinary differential equations.

II. METHODS AND MATERIAL

Consider the forth order initial value problem (IVP) of the form

$$y^{(4)} + a(x)y''' + b(x)y'' + c(x)y' + d(x)y = f(x), \qquad 0 \le x \le L$$

with the initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta, \qquad y''(0) = \gamma \quad \text{and } y'''(0) = \delta$$

……………..(2)

where a(x), b(x), c(x) and d(x) are continuous functions with the given constant α, β, γ *and* δ. To reduce the forth order equation (1) into first order system of ordinary differential equation, let use the substations z(x) = y'(x), u(x) = y''(x) and v(x) = y'''(x), then the given initial value problem of equation (1) with equation (2) can be re-written as:

$$\begin{cases} y' = z, \ y(0) = \alpha \\ y'' = z' = u, \ y'(0) = z(0) = \beta \\ y''' = u' = v, \ y'''(0) = u(0) = y \\ y^{(4)} = v', \ y'''(0) = v(0) = \delta \end{cases}$$

……………………..(3)

Thus, we will obtain the system which written in form:

$$\begin{cases} z = f_1(x, y, z, u, v), \ y(0) = \alpha \\ z' = u = f_2(x, y, z, u, v), \ z(0) = \beta \\ u' = v = f_3(x, y, z, u, v), \ u(0) = \gamma \\ v' = f_4(x, y, z, u, v), \ v(0) = \delta \end{cases}$$

………………………..(4)

where g(x, y, z, u, v) = f(x) – a(x)v – b(x)u – c(x)z – d(x)y

Dividing the interval [0, L] into N equal subinterval of mesh length h and the mesh point is given by $x_i = x_0 + ih$ for i = 1, 2, ….., N-1. For the sake of the simplicity let use the denotation $y(x_i) = y$, $z(x_i) = z$, $u(x_i) = u$ and $v(x_i) = v$ etc.

To solve the system of the initial value problems written in equation (4), we apply the single step methods that require information about the solution at x_i to calculate x_{i+1}, (Grewal 2002, Jain et al. 2007). From the single step methods and the family of Runge-Kutta methods, the general numerical solution of Eq. (4) using fifth Runge-Kutta method given as:

$$\begin{cases} k_i = hf_1\left(x_i + c_i h, \ \mathrm{y_i} + \sum_{j=1}^{4} a_{ij} k_j, \ \mathrm{z_i} + \sum_{j=1}^{4} a_{ij} m_j, \mathrm{u_i} + \sum_{j=1}^{4} a_{ij} k_j, \mathrm{v_i} + \sum_{j=1}^{4} a_{ij} m_j, \right) \\ L_i = hf_2\left(x_i + c_i h, \ \mathrm{y_i} + \sum_{j=1}^{4} a_{ij} k_j, \ \mathrm{z_i} + \sum_{j=1}^{4} a_{ij} m_j, \mathrm{u_i} + \sum_{j=1}^{4} a_{ij} k_j, \mathrm{v_i} + \sum_{j=1}^{4} a_{ij} m_j, \right) \\ m_i = hf_3\left(x_i + c_i h, \ \mathrm{y_i} + \sum_{j=1}^{4} a_{ij} k_j, \ \mathrm{z_i} + \sum_{j=1}^{4} a_{ij} m_j, \mathrm{u_i} + \sum_{j=1}^{4} a_{ij} k_j, \mathrm{v_i} + \sum_{j=1}^{4} a_{ij} m_j, \right) \\ q_i = hf_4\left(x_i + c_i h, \ \mathrm{y_i} + \sum_{j1}^{4} a_{ij} k_j, \ \mathrm{z_i} + \sum_{j=1}^{4} a_{ij} m_j, \mathrm{u_i} + \sum_{j=1}^{4} a_{ij} k_j, \mathrm{v_i} + \sum_{j=1}^{4} a_{ij} m_j, \right) \end{cases}$$

Nikolaos Christodoulou, 2009 was present the fifth order Runge-Kutta method to solve a first order initial value problem of the form dy/dt = f(x, y), y(x0) = y0, which is given by the following equation:

$$y_{n+1} = y_n + \frac{7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6}{90} \qquad \text{.........(6)}$$

where $k_1 = hf\left(x_n, y_n\right)$, $k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$, $k_3 = hf\left(x_n + \frac{h}{4}, y_n + \frac{3k_1 + k_2}{16}\right)$

$$k_4 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_3}{2}\right), \quad k_5 = hf\left(x_n + \frac{3h}{4}, y_n + \frac{-3k_2 + 6k_3 + 9k_4}{16}\right)$$

$$k_6 = hf\left(x_n + h, y_n + \frac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7}\right)$$

Thus, to solve the system of initial value problem of Eq.(4), one of the families of the fifth order Runge-Kutta method can be re-written as: ....(7)

where

$$k_1 = f_1\left(x_n, y_{n,} z_{n,} u_{n,} v_n\right)$$

$$L_1 = f_2\left(x_n, y_{n,} z_{n,} u_{n,} v_n\right)$$

$$m_1 = f_3\left(x_n, y_{n,} z_{n,} u_{n,} v_n\right)$$

$$q_1 = g\left(x_n, y_{n,} z_{n,} u_{n,} v_n\right)$$

$$k_2 = hf_1\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{L_1}{2},\ u_n + \frac{m_1}{2},\ v_n + \frac{q_1}{2}\right)$$

$$L_2 = hf_1\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{L_1}{2},\ u_n + \frac{m_1}{2},\ v_n + \frac{q_1}{2}\right)$$

$$m_2 = hf_1\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{L_1}{2},\ u_n + \frac{m_1}{2},\ v_n + \frac{q_1}{2}\right)$$

$$q_2 = hg\left(x_n + \frac{h}{2},\ y_n + \frac{k_1}{2},\ z_n + \frac{L_1}{2},\ u_n + \frac{m_1}{2},\ v_n + \frac{q_1}{2}\right)$$

$$k_3 = hf_1\left(x_n + \frac{h}{4},\ y_n + \frac{3k_1 + k_2}{16},\ z_n + \frac{3L_1 + L_2}{16},\ u_n + \frac{3m_1 + m_2}{16},\ v_n + \frac{3q_1 + q_2}{16}\right)$$

$$L_3 = hf_1\left(x_n + \frac{h}{4},\ y_n + \frac{3k_1 + k_2}{16},\ z_n + \frac{3L_1 + L_2}{16},\ u_n + \frac{3m_1 + m_2}{16},\ v_n + \frac{3q_1 + q_2}{16}\right)$$

$$m_3 = hf_1\left(x_n + \frac{h}{4},\ y_n + \frac{3k_1 + k_2}{16},\ z_n + \frac{3L_1 + L_2}{16},\ u_n + \frac{3m_1 + m_2}{16},\ v_n + \frac{3q_1 + q_2}{16}\right)$$

$$q_3 = hg\left(x_n + \frac{h}{4},\ y_n + \frac{3k_1 + k_2}{16},\ z_n + \frac{3L_1 + L_2}{16},\ u_n + \frac{3m_1 + m_2}{16},\ v_n + \frac{3q_1 + q_2}{16}\right)$$

$$k_4 = hf_1\left(x_n + \frac{h}{2},\ y_n + \frac{k_3}{2},\ z_n + \frac{L_3}{2},\ u_n + \frac{m_3}{2},\ v_n + \frac{q_3}{2}\right)$$

$$L_4 = hf_1\left(x_n + \frac{h}{2},\ y_n + \frac{k_3}{2},\ z_n + \frac{L_3}{2},\ u_n + \frac{m_3}{2},\ v_n + \frac{q_3}{2}\right)$$

$$m_4 = hf_1\left(x_n + \frac{h}{2},\ \mathrm{y_n} + \frac{k_3}{2},\ \mathrm{z_n} + \frac{L_3}{2},\ \mathrm{u_n} + \frac{m_3}{2},\ \mathrm{v_n} + \frac{q_3}{2} \right)$$

$$q_4 = hg\left(x_n + \frac{h}{2},\ \mathrm{y_n} + \frac{k_3}{2},\ \mathrm{z_n} + \frac{L_3}{2},\ \mathrm{u_n} + \frac{m_3}{2},\ \mathrm{v_n} + \frac{q_3}{2} \right)$$

$$k_5 = hf_1\left(\begin{array}{l} x_n + \dfrac{3h}{4},\ \mathrm{y_n} + \dfrac{-3k_2 + 6k_3 + 9k_4}{16},\ \mathrm{z_n} + \dfrac{-3L_2 + 6L_3 + 9L_4}{16},\ \mathrm{u_n} + \dfrac{-3m_2 + 6m_3 + 9m_4}{16}, \\ \mathrm{v_n} + \dfrac{-3q_2 + 6q_3 + 9q_4}{16} \end{array} \right)$$

$$L_5 = hf_1\left(\begin{array}{l} x_n + \dfrac{3h}{4},\ \mathrm{y_n} + \dfrac{-3k_2 + 6k_3 + 9k_4}{16},\ \mathrm{z_n} + \dfrac{-3L_2 + 6L_3 + 9L_4}{16},\ \mathrm{u_n} + \dfrac{-3m_2 + 6m_3 + 9m_4}{16}, \\ \mathrm{v_n} + \dfrac{-3q_2 + 6q_3 + 9q_4}{16} \end{array} \right)$$

$$m_5 = hf_1\left(\begin{array}{l} x_n + \dfrac{3h}{4},\ \mathrm{y_n} + \dfrac{-3k_2 + 6k_3 + 9k_4}{16},\ \mathrm{z_n} + \dfrac{-3L_2 + 6L_3 + 9L_4}{16},\ \mathrm{u_n} + \dfrac{-3m_2 + 6m_3 + 9m_4}{16}, \\ \mathrm{v_n} + \dfrac{-3q_2 + 6q_3 + 9q_4}{16} \end{array} \right)$$

$$q_5 = hg\left(\begin{array}{l} x_n + \dfrac{3h}{4},\ \mathrm{y_n} + \dfrac{-3k_2 + 6k_3 + 9k_4}{16},\ \mathrm{z_n} + \dfrac{-3L_2 + 6L_3 + 9L_4}{16},\ \mathrm{u_n} + \dfrac{-3m_2 + 6m_3 + 9m_4}{16}, \\ \mathrm{v_n} + \dfrac{-3q_2 + 6q_3 + 9q_4}{16} \end{array} \right)$$

$$k_6 = hf_1\left(\begin{array}{l} x_n + h,\ \mathrm{y_n} + \dfrac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7},\ \mathrm{z_n} + \dfrac{L_1 + 4L_2 + 6L_3 - 12L_4 + 8L_5}{7}, \\ \mathrm{u_n} + \dfrac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7},\ \mathrm{v_n} + \dfrac{q_1 + 4q_2 + 6q_3 - 12q_4 + 8q_5}{7} \end{array} \right)$$

$$L_6 = hf_1\left(\begin{array}{l} x_n + h,\ \mathrm{y_n} + \dfrac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7},\ \mathrm{z_n} + \dfrac{L_1 + 4L_2 + 6L_3 - 12L_4 + 8L_5}{7}, \\ \mathrm{u_n} + \dfrac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7},\ \mathrm{v_n} + \dfrac{q_1 + 4q_2 + 6q_3 - 12q_4 + 8q_5}{7} \end{array} \right)$$

$$m_6 = hf_1\left(\begin{array}{l} x_n + h,\ \mathrm{y_n} + \dfrac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7},\ \mathrm{z_n} + \dfrac{L_1 + 4L_2 + 6L_3 - 12L_4 + 8L_5}{7}, \\ \mathrm{u_n} + \dfrac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7},\ \mathrm{v_n} + \dfrac{q_1 + 4q_2 + 6q_3 - 12q_4 + 8q_5}{7} \end{array} \right)$$

$$q_6 = hg\left(\begin{array}{l} x_n + h,\ \mathrm{y_n} + \dfrac{k_1 + 4k_2 + 6k_3 - 12k_4 + 8k_5}{7},\ \mathrm{z_n} + \dfrac{L_1 + 4L_2 + 6L_3 - 12L_4 + 8L_5}{7}, \\ \mathrm{u_n} + \dfrac{m_1 + 4m_2 + 6m_3 - 12m_4 + 8m_5}{7},\ \mathrm{v_n} + \dfrac{q_1 + 4q_2 + 6q_3 - 12q_4 + 8q_5}{7} \end{array} \right)$$

In the determination of the parameters, since the term are up to $O(h^5)$ be compared, the truncation error is $O(h^6)$ and the order of the method is $O(h^5)$ (Akanbi, 2010 and Shampine and watts, 1971).

Numerical Examples and Results:

To validate the applicability of the methods is three examples with initial conditions have been considered. For each number of nodal points N, the point wise absolute error are approximated by the formula $\|E\| = |y(x_i) - y_i|$, for I = 0, 1, 2,……..N, where $y(x_i)$ and y_i are the exact and computed approximate solution of the given problem respectively, at the nodal point x_i. Numerical examples are given to illustrate and convergence of the method.

Example 1. Consider following forth order initial value problem:

$$y^{iv} = (x^4 + 14x^3 + 49x^2 + 32x - 2)e^x, \qquad y(0) = y'(0) = 0,\ y''(0) = 2,\ y'''(0) = -6,\ 0 \le x \le 1$$

Exact solution: $y(x) = e^x$

Table 1: Numerical solution and comparisons of Absolute error for example 1

x	Exact Solution	Numerical Solution	Absolute Error	Relative error
0.1	0.008951884436413	0.008951946802962	6.23665E-08	6.96681E-06
0.2	0.031267910608900	0.031220727000262	4.71836E-05	0.001511291
0.3	0.059528773414102	0.059572783069585	4.40097E-05	0.000738754
0.4	0.085929102584137	0.085852298979918	7.68036E-05	0.000894602
0.5	0.103045079418758	0.103262809094288	0.00021773	0.0021085
0.6	0.104954042902493	0.107491344276248	0.002537301	0.023604704
0.7	0.088806494399448	0.096583671275298	0.007777177	0.080522688
0.8	0.056973847769407	0.065804004352781	0.008830157	0.134188742
0.9	0.019922785200371	0.018401665472288	0.00152112	0.082662068
1	0.0	0.0	0	0

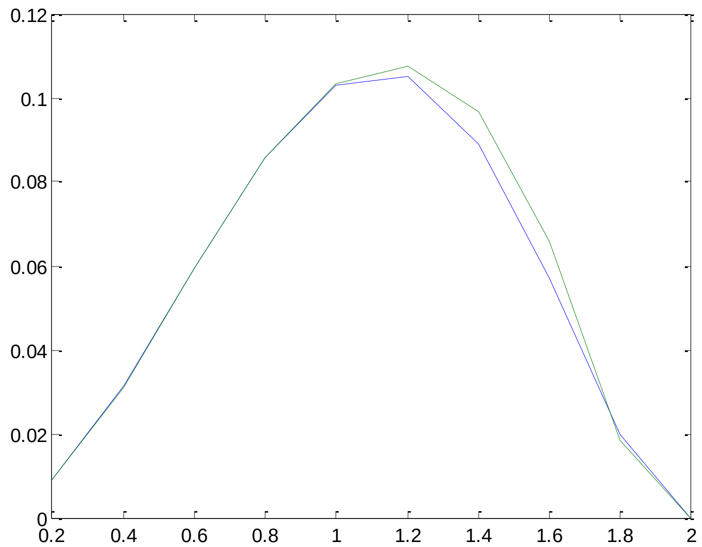

Fig. 1

Example 2. Consider following forth order initial value problem:

$$y^{iv} + y'' = 0, \qquad y(0) = 0,\ \ y'(0) = \frac{-1.1}{72 - 50\pi},\ y''(0) = \frac{1}{144 - 100\pi},\ y'''(0) = \frac{1.2}{144 - 100\pi},\ \ 0 \le x \le \frac{\pi}{2}$$

Exact solution: $y(x) = \dfrac{1 - x - \cos x - 1.2\sin x}{144 - 100\pi}$

Table 2 : Numerical solution and comparisons of Absolute error for example 2

x	Exact Solution	Numerical Solution	Absolute Error	Relative error
0	0	0	0	0
π/20	0.001953988991557	0.001837620944741	0.000116368	0.063325381
π/10	0.003737887405439	0.003585077615441	0.00015281	0.042623844
3π/20	0.005330500340377	0.005209665055475	0.000120835	0.023194444
4π/20	0.006715342978434	0.006673584119669	4.17589E-05	0.006257336
5π/20	0.007881046495933	0.007933188340953	5.21418E-05	0.006572622
6π/20	0.008821638001625	0.008938171207836	0.000116533	0.013037701
7π/20	0.009536687609097	0.009749178475500	0.000212491	0.021795772
8π/20	0.010031319022000	0.010073419464246	4.21004E-05	0.00417936
9π/20	0.010316083371461	0.010310711228363	5.37214E-06	0.000521025
10π/20	0.010406699412219	0.010406240600001	4.58812E-07	4.40901E-05

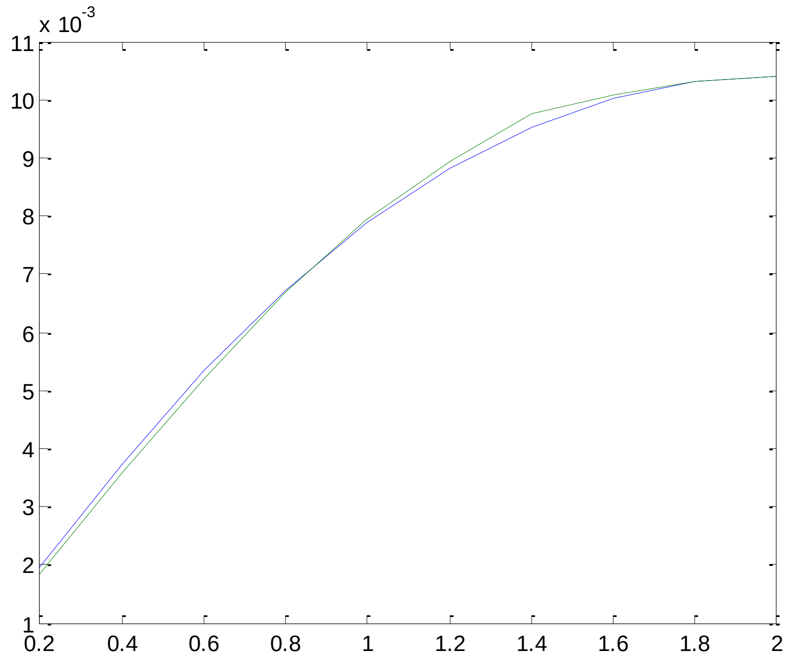

Fig. 2

Example 3. Consider following forth order initial value problem:

$$y^{iv} = x, \qquad y(0) = y'(0) = 1,\ y''(0) = 2,\ y'''(0) = 0, \qquad 0 \le x \le 1$$

Exact solution: $y(x) = \dfrac{x^5}{120} + x$

x	Exact Solution	Numerical Solution	Absolute error	Relative error
0	0	0	0	0
0.1	0.100000083333333	0.10000008333333	2.9976E-15	2.9976E-14
0.2	0.200002666666667	0.20000266666667	2.9976E-15	1.49878E-14
0.3	0.300002025000000	0.30002025000000	1.8225E-05	6.07459E-05
0.4	0.400008533333333	0.40008533333333	7.68E-05	0.000191959
0.5	0.500260416666667	0.50026041666666	6.99441E-15	1.39815E-14
0.6	0.600648000000000	0.60064800000000	0	0
0.7	0.701400583333333	0.70140058333333	2.9976E-15	4.27374E-15
0.8	0.802730666666667	0.80273066666667	2.9976E-15	3.73426E-15
0.9	0.904920750000000	0.90492075000000	0	0
1.0	1.000833333333334	1.008333333333334	0.0075	0.007438017

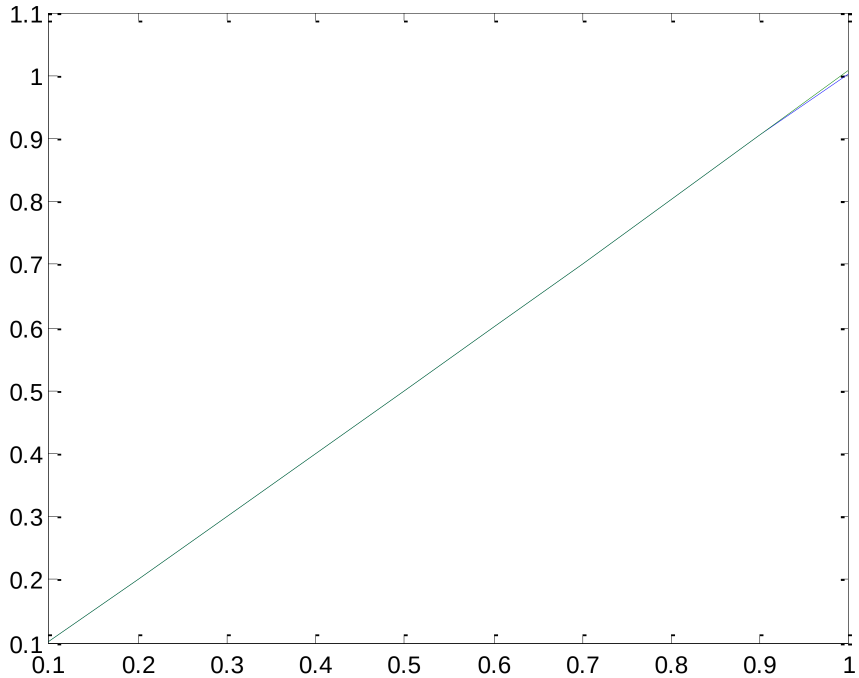

Fig. 3

III. CONCLUSION

On the basis of the above discussion we get the result obtained by analytical methods is always providing accurate solution but numerical solution always providing approximate result. To validate the applicability of the proposed method, three model examples have been considered and solve for different value of x. We observed from numerical solution presented in table (1-3) and graph(1-3), Runge-kutta fifth order method approximates the exact solution very well. The accuracy of present method is better than the other existing method. Thus, the fifth order Runge-Kutta method is more accurate and a preferable method to find the approximate solution of the forth order initial value problems.

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