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A Study of Infected Prey with Predator - Prey Model

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ABSTRACT

Incidence rate plays a main role in the modeling of epidemic dynamics. In this paper we discussed two preys and one predator model. We define prey predator model with susceptible and infected prey. The stability of the system discussed with equilibrium points. Numerical simulation has been also performed in support of analysis. Also for different values of constant parameters the equilibrium level has been tabulated.

Keywords : Prey-Predator Model, SI Models, Population Dynamics, Epidemic, Stability.

I. INTRODUCTION

Along with all Mathematical models, prey-predator models have received much attention during the last few decades due to its wide range of application. The dynamics relationship between predators and their prey have long been one of the dominant themes due to its universal prevalence and importance. The prey-predator interaction has been described firstly by two pioneers Lotka [11] and Volterra [16] in two independent works. After them, more realistic prey predator model was introduced by Holling [8]. In the subsequent time many authors proposed and studied different predator-prey models in the presence of disease. Chattopadhyay and Arino [4] coined the name eco-epidemiology for the study of such systems.

The Italian Mathematician Volterra [16] proposed a differential equation model to describe the population dynamics of two interact species, a predator and its prey. Such Mathematical models become useful in describing how populations vary over time. Data about the various rates of growth, death, and relations of species naturally lead to models involving differential equations.

Eco-epidemiology is one of the branches in Mathematical biology which considers both the ecological and epidemiological issues simultaneously. Eco-epidemiology proved the direct and indirect effects that diseases have on interacting populations by Venturino [15]. The initial effects of demographical changes, namely population growth, have been introduced in epidemic models Gao [6], Mena [12]. The epidemics have been considered in models of interact populations in different environments by various researchers [1,3,5,7,13,14].

Infectious diseases have been known to be an important regulating factor for human and animal population sizes. Hsieh and Hsiao [9] reviewed that many ecological generalize of predator–prey systems with disease, it is reported that the predators take an excessively high number of parasite of infected prey. The models for the spread of infectious diseases originated from the classic work of Kermack and McKendrick [10]. After these original works in two different fields, lot of research work has been done both in theoretical ecology and epidemiology. Anderson and May [2] were the first who combined the above two fields and formulated a predator-prey model where prey species be infected by some diseases.

In this paper we discuss the dynamics of a communicable disease in a predator–prey model with disease infection prey. It is assumed that a parasite is infectious and spreads among preys according to a SI (susceptible-infective) model. The presence of both preys has affect predator populations. A small thought has been addressed to the merge of these two important areas, ecology and epidemiology.

II. DISCRIPTION OF THE MODEL

We study a prey predator relationship with existence of infected prey. In this paper we assumed that the interaction between the two classes of prey follow the SI model and another side we assumed the predator will only interact with the unhealthy prey.

A Mathematical model is proposed and analyzed by Wuhaib and Hasan [17] in the following differential equation.

$$X'(t) = rX\left(1-\frac{X}{K}\right)-\frac{PXYK}{\alpha X+K},$$
$$Y'(t) = \frac{PXYK}{\alpha X+K}-\frac{\gamma YZ}{Z+\gamma\beta Y}, \qquad (2.1)$$
$$Z'(t) = \frac{e\gamma YZ}{Z+\gamma\beta Y}-dZ.$$

Here X is the susceptible prey population, Y is the infected prey, Z is the predator, r is the growth rate, K is the carrying capacity, P is the incidence rate, λ is the total attack rate for predator, β is the handling time, e is the conversion efficiency and d *is* the death rate of predator, α is a constant such that $0.4 \le \alpha \le 1.0$, $\frac{\gamma YZ}{Z+\gamma\beta Y}, \frac{e\gamma YZ}{Z+\gamma\beta Y}$ are the Michaelis-Mention–Holling [8] functional and numerical responses. Now we reduce the number of parameters by letting

$$x=\frac{X}{K}, y=\frac{Y}{K}, z=\frac{Z}{\gamma\beta K}$$

We suppose that $k=\frac{PK}{r}, b=\frac{\gamma}{r}, c=\frac{e}{r\beta}, a=\frac{d}{r}$

Then the system of equations (2.1) can be written as

$$x'(t)=x(1-x)-\frac{kxy}{\alpha x+1},$$
$$y'(t)=\frac{kxy}{\alpha x+1}-b\frac{yz}{z+y}, \qquad (2.2)$$
$$z'(t)=c\frac{yz}{z+y}-az.$$

Theorem: The solution of the system (2.2) is bounded.
Proof: We define $w(t)=x(t)+y(t)+z(t)$ and let μ any positive number such that $a>\mu$ then

$$\dot{w}+\mu w=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dz}{dt}+\mu x+\mu y+\mu z$$

$$\dot{w}+\mu w=-x^2+(1+\mu)x-(b-c)\frac{yz}{y+z}-(a-\mu)z+\mu y$$

If we take $b>c$ and assume the unhealthy prey is comparable to the healthy prey, we get

$$\dot{w}+\mu w\le-\left(x^2+(1+2\mu)x+\frac{(1+2\mu)}{2}\right)^2+\left(\frac{(1+2\mu)}{2}\right)^2$$

$$\dot{w}+\mu w\le-\left(x+\frac{(1+2\mu)}{2}\right)^2+\left(\frac{(1+2\mu)}{2}\right)^2$$

$$\dot{w}+\mu w\le+\left(\frac{(1+2\mu)}{2}\right)^2$$

Suppose that $\left(\frac{(1+2\mu)}{2}\right)^2=v$

$$\dot{w}+\mu w\le v$$

Thus

$$0<w(x.y,z)\le\frac{v}{u}\left(1-e^{-ut}\right)+e^{-ut}\left(x.y,z\right)_{t=0}.$$

III. EXISTENCE OF THE EQUILIBRIA

Equilibrium of the equations (2.2) can be obtained by making right hand side equal to zero. This provides three equilibrium $E_0(0,0,0), E_1(1,0,0), \hat{E}(\hat{x},\hat{y},\hat{z})$. The equilibrium E_0 and E_1 is trivial.

$$x(1-x)-\frac{kxy}{\alpha x+1}=0,$$
$$\frac{kxy}{\alpha x+1}-b\frac{yz}{z+y}=0,$$
$$c\frac{yz}{z+y}-az=0.$$

Equilibrium point $\hat{E}(\hat{x},\hat{y},\hat{z})$ of the system (2.2) exists

$$\hat{x}=\frac{b\left(1-\frac{a}{c}\right)}{k-b\left(1-\frac{a}{c}\right)\alpha},$$

$$\hat{y}=\frac{(\alpha\hat{x}+1)(1-\hat{x})}{k},$$

$$\hat{z}=\hat{y}\frac{c}{a}\left(1-\frac{a}{c}\right).$$

$\hat{x}$ and $\hat{z}$ be positive, we must have $c>a$. The variation matrix of system (2.2) is given by

$$A=\begin{bmatrix} 1-2x-\frac{ky}{(\alpha x+1)^2} & \frac{-kx}{(\alpha x+1)} & 0 \\ \frac{ky}{(\alpha x+1)^2} & \frac{kx}{(\alpha x+1)}-\frac{bz^2}{(z+y)^2} & -\frac{by^2}{(z+y)^2} \\ 0 & \frac{cz^2}{(y+z)^2} & \frac{cz^2}{(y+z)^2}-a \end{bmatrix}$$

The $J_0(0,0,0)$ and $J_1(1,0,0)$ gives by characteristics equation

$$|A-\lambda I|=0.$$

Then J_0 and J_1 is not asymptotically stable because the second Eigen value has zero real part and second Eigen value is positive, respectively.

We discuss the stability of $\hat{E}(\hat{x},\hat{y},\hat{z})$ in this form

$$\hat{J}=\begin{bmatrix} 1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1} & \frac{-k\hat{x}}{\alpha\hat{x}+1} & 0 \\ \frac{1-\hat{x}}{\alpha\hat{x}+1} & \frac{k\hat{x}}{\alpha\hat{x}+1}-b\left(1-\frac{a}{c}\right)^2 & \frac{-a^2b}{c^2} \\ 0 & c\left(1-\frac{a}{c}\right)^2 & -a\left(1-\frac{a}{c}\right) \end{bmatrix}$$

The characteristic polynomial of $\hat{J}$ is.

$\lambda^3-A\lambda^2+B\lambda-C=0$.

Where

$$-A=-(a_{11}+a_{22}+a_{33})$$

$$A=-\left[\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)+\left(\frac{k\hat{x}}{\alpha\hat{x}+1}-b\left(1-\frac{a}{c}\right)^2\right)-a\left(1-\frac{a}{c}\right)\right].$$

$A>0$ if $c>a$

$$B=a_{11}a_{33}+a_{22}a_{33}+a_{11}a_{22}-a_{23}a_{32}-a_{12}a_{21}$$

$$-\left[\begin{aligned}&\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(a\left(1-\frac{a}{c}\right)\right)+\left(\frac{k\hat{x}}{\alpha\hat{x}+1}-b\left(1-\frac{a}{c}\right)^2\right)\left(a\left(1-\frac{a}{c}\right)\right)\\&+\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(b\left(1-\frac{a}{c}\right)^2-\frac{k\hat{x}}{\alpha\hat{x}+1}\right)-\left(\frac{a^2b}{c^2}\right)\left(c\left(1-\frac{a}{c}\right)^2\right)\\&-\left(\frac{k\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\end{aligned}\right].$$

and

$$C=-a_{11}a_{22}a_{33}+a_{12}a_{21}a_{33}+a_{11}a_{23}a_{32}$$

$$C=\left\{\begin{aligned}&-\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(b\left(1-\frac{a}{c}\right)-\frac{k\hat{x}}{\alpha\hat{x}+1}\right)\left(a\left(1-\frac{a}{c}\right)\right)\\&+\left(\frac{-k\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(-a\left(1-\frac{a}{c}\right)\right)\\&+\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{a^2b}{c^2}\right)\left(c\left(1-\frac{a}{c}\right)^2\right)\end{aligned}\right\}$$

AB-C=

$$\left\{\left[\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)+\left(\frac{k\hat{x}}{\alpha\hat{x}+1}-b\left(1-\frac{a}{c}\right)^2\right)-a\left(1-\frac{a}{c}\right)\right]\right\}$$

$$\left\{\begin{aligned}&\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(a\left(1-\frac{a}{c}\right)\right)+\left(\frac{k\hat{x}}{\alpha\hat{x}+1}-b\left(1-\frac{a}{c}\right)^2\right)\left(a\left(1-\frac{a}{c}\right)\right)\\&+\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(b\left(1-\frac{a}{c}\right)^2-\frac{k\hat{x}}{\alpha\hat{x}+1}\right)-\left(\frac{a^2b}{c^2}\right)\left(c\left(1-\frac{a}{c}\right)^2\right)\\&-\left(\frac{k\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\end{aligned}\right\}.$$

$$-\left\{\begin{aligned}&-\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(b\left(1-\frac{a}{c}\right)^2-\frac{k\hat{x}}{\alpha\hat{x}+1}\right)\left(a\left(1-\frac{a}{c}\right)\right)\\&+\left(\frac{k\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(a\left(1-\frac{a}{c}\right)\right)\\&+\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)\left(\frac{a^2b}{c^2}\right)\left(c\left(1-\frac{a}{c}\right)^2\right)\end{aligned}\right\}$$

$AB-C>0$, $\left(1-2\hat{x}-\frac{1-\hat{x}}{\alpha\hat{x}+1}\right)>0$ and $c<a.$

IV. NUMERICAL SIMULATION AND CONCLUSION

Using the parameters of $a = 0.4, b = 0.8, c = 0.5, k = 1.5,$ with the different values of such that $0.4 \le \alpha \le 1.0.$ For the values of parameters we find the equilibrium points $(\hat{x}, \hat{y}, \hat{z}).$ The following table shows the variation of $(\hat{x}, \hat{y}, \hat{z})$ its analysis is performed to study the effect of constant α and we get the optimal solution.

α	a	b	c	k	$\hat{x}$	$\hat{y}$	$\hat{z}$
1.0	0.3	0.8	0.5	1.5	0.2711	0.6176	0.4116
0.9	0.3	0.8	0.5	1.5	0.2640	0.6072	0.4047
0.8	0.3	0.8	0.5	1.5	0.2572	0.5970	0.3979
0.7	0.3	0.8	0.5	1.5	0.2507	0.5871	0.3913
0.6	0.3	0.8	0.5	1.5	0.2446	0.5775	0.3849
0.5	0.3	0.8	0.5	1.5	0.2388	0.5680	0.3784
0.4	0.3	0.8	0.5	1.5	0.2332	0.5588	0.3724

Thus we observe that the equilibrium points $(\hat{x}, \hat{y}, \hat{z})$ are going down with α and we get stability of the model.

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