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Orthogonal Series' of Absolute Banach Summability

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ABSTRACT

In this paper we have proved a theorem on "*Orthogonal Series*'of *Absolute Banach Summability"* which generalizes known result. However our theorem is as follows.

Theorem: Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(\mathrm{n})}$ converges and $\int_0^{\pi}\frac{d\phi_n(t)}{t^{\upsilon}}<\infty$

then the orthogonal series $\sum_{n=1}^{\infty}a_n\,\phi_n\,(x)$ is $|B|$ summable at $t=x$, provided

$$\sum_k k^{\beta-1}(n+k)=O\big(n\,\Omega(n)\big),$$

Where $0<\beta<\gamma<1$.

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I. DEFINITIONS AND NOTATIONS

1. Let $\{S_n\}$ be the sequence of partial sums of a series $\sum a_n$. Let the sequence $\{t_k\,(n)\}_{k=1}^{\infty}$ is defined by

(1.1) $t_k(n)=\frac{1}{k}\sum_{\upsilon=0}^{k-1}s_{n+\upsilon}k\in N$ if

(1.2) $\lim_{k\to\infty}\mathrm{t_k(n)}=\mathrm{s}$ a finite number,

Uniformly for all $n\in N$, then $\sum u_n$ is said to be Banach summability to s.

Further if,

(1.3) $$\sum_{k=1}^{\infty}|t_k(n)-t_{k+1}(n)|<\infty$$

Uniformly for all $n\in N$, then the series $\sum u_n$ is said to be absolute Banach summable or simply $|B|$ −summable.

2. Let $\{\phi_n(x)\}$ be an orthogonal system defined in the interval(a,b). We suppose that $f(x)$ belongs to $2!\,(a,b)$ and $f(x)\sim\sum_{n=0}^{\infty}a_n\,\phi_n\,(x)$ by $E_n^{(2)}(f)$, we denote the best approximation to $f(x)$ in the metric of $2!$ by means of polynomials

$$\sum_{k=0}^{n-1}a_k\phi_k(x)\quad \text{i.e.}\quad \left\{E_n^{(2)}(f)\right\}^2=\sum_{k=n}^{\infty}|a_k|^2$$

We write $\Delta\lambda_n=\lambda_n-\lambda_{n-1}$

((2.1) $$g(k,t)=\frac{2}{\pi}\frac{1}{k(k+1)}\sum_{\upsilon=1}^{k}\frac{\upsilon}{(n+\upsilon)}(n+\upsilon)^{\upsilon-\beta}\frac{\Omega(t)}{t^2}$$

((2.2) $$J(k,u)=\frac{1}{F(1-\beta)}\int_u^{\pi}\frac{d}{dt}g(k,t)\,(t-u)^{-\beta}\,dt$$

$$\omega(k,u)=u^{\upsilon}J(k,u)$$

$[x]$ = greatest integer not exceeding x

$$U = \left[\frac{1}{u}\right] \text{ and } \tau = \left[\frac{1}{t}\right]$$

II. INTRODUTION

Ul'yanov [7] has proved the following theorems on $|C,\alpha|$ summability.

Theorem A:

If $1 \geq \alpha \geq \frac{1}{2}$ and $\sum_{n=n_0}^{\infty} |a_n|^2 \log_n (\log\log n)^{1+\varepsilon}$ converges, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is summable $|C,\alpha|$ almost everywhere.

Theorem B:

If $0 < \alpha < \frac{1}{2}$ and $\sum_{n=n_0}^{\infty} |a_n|^2 n^{1-2\alpha} \log_n (\log n)^{1+\varepsilon}$ converges, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is summable $|C,\alpha|$ almost everywhere.

Theorem C:

If $1 \geq \alpha \geq \frac{1}{2}$ and $\sum_{n=n_0}^{\infty} n^{-1}(\log\log n)^{1+\varepsilon}\left\{E_n^{(2)}(f)\right\}^2$ converges, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is summable $|C,\alpha|$ almost everywhere.

Theorem D:

If $0 < \alpha < \frac{1}{2}$ and $\sum_{n=n_0}^{\infty} n^{-2\alpha} \log_{\mathrm{n}}(\log\log n)^{1+\varepsilon}\left\{E_n^{(2)}(f)\right\}^2$ converges, then the series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is summable $|C,\alpha|$ almost everywhere.

Generalizing the above theorems Okuyama [6] has proved the following theorem for $|N,p_n|$ summability of orthogonal series.

Theorem E:

Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(\mathrm{n})}$ converges. Let $\{p_n\}$ be non-negative and non-increasing. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n) w_n$ converges, then the orthogonal series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is summable $|N,p_n|$ almost everywhere. Where

$$\omega_k = \frac{1}{k}\sum_{n=k}^{\infty} \frac{n^2 p_n^2 - p_{n-k}^2}{p_n^{\Delta}}\left(\frac{P_n}{p_n} - \frac{P_{n-k}}{p_{n-k}}\right)^2$$

The main object of this paper is to generalize Theorem E, for orthogonal series of absolute Banach summability. We establish our result in the form of following theorem

Theorem:

Let $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(\mathrm{n})}$ converges and

$$\int_0^{\pi} \frac{d\phi_n(t)}{t^{\upsilon}} < \infty$$

then the orthogonal series $\sum_{n=1}^{\infty} a_n \phi_n(x)$ is $|B|$ summable at $t = x$, provided

(3.1) $$\sum_k k^{\beta-1}(n+k) = O\big(n\,\Omega(n)\big),$$

Where $0 < \beta < \gamma < 1$.

III. PROOF OF THE THEOREM:

In order to prove the theorem, we have to prove that

$$\sum_{k=1}^{\infty} |t_k(n) - t_{k+1}(n)| = O(1)$$

Now taking,

$$t_k(n) - t_{k+1}(n) = \frac{1}{k(k+1)} \sum_{\upsilon=1}^{k} \upsilon(n+\upsilon)^{\gamma-\beta} \phi_{n+\upsilon}(t)$$

$$= \frac{1}{k(k+1)} \sum_{\upsilon=1}^{k} \upsilon(n+\upsilon)^{\gamma-\beta} \frac{2}{\pi} \int_0^\pi \frac{\phi_n(t)\Omega(t)}{t} dt$$

$$= t \frac{1}{k(k+1)} \sum_{\upsilon=1}^{k} \frac{\upsilon}{(n+\upsilon)} (n+\upsilon)^{\gamma-\beta} \frac{2}{\pi} \int_0^\pi \phi_n(t) \frac{\Omega(t)}{t^2} dt$$

$$= \int_0^\pi \phi_n(t) \left[\frac{2}{\pi} \frac{1}{k(k+1)} \sum_{\upsilon=1}^{k} \frac{\upsilon}{(n+\upsilon)} (n+\upsilon)^{\gamma-\beta} \frac{\Omega(t)}{t^2} \right] dt$$

$$0 < \beta < \gamma < 1 \Rightarrow 0 < r - \beta < 1$$

$$= \int_0^\pi \phi_n(t) \frac{d}{dt} g(k,t)\, dt$$

$$= \int_0^\pi \frac{d}{dt} g(k,t) \left\{ \frac{1}{\Gamma(1-\beta)} \int_0^t (t-u)^{-\beta}\, d\phi_\beta(u) \right\} dt$$

$$= \frac{1}{\Gamma(1-\beta)} \int_0^\pi d\phi_\beta(u) \int_u^\pi \frac{d}{dt} g(k,t)(t-u)^\beta\, dt$$

$$= \int_0^\pi d\phi_\beta(u) \left\{ \frac{1}{\Gamma(1-\beta)} \int_u^\pi \frac{d}{dt} g(k,t)(t-u)^\beta\, dt \right\}$$

$$= \int_0^\pi J(u,u) d\phi_\beta(u)$$

$$= \int_0^\pi u^\upsilon J(k,k) \frac{d\phi_\beta(u)}{u^\upsilon}$$

$$= \int_0^\pi \omega(k,u)\, \frac{d\phi_\beta(u)}{u^\upsilon}$$

Now,

$$\sum_{k=1}^{\infty} |t_k(n) - t_{k+1}(n)| = \int_0^u \sum_{k=1}^{\infty} |\omega(k,u)|\, \frac{d\phi_\beta(u)}{u^\upsilon}$$

since

$\int_0^\pi \frac{d\phi_\beta(u)}{u^\upsilon} < \infty$, then the theorem is proved

$\sum_{k=1}^{\infty} |\omega(k,u)| < \infty$, uniformly for all u.

Now,

$$\sum_{k=1}^{\infty} |\omega(k,u)| = \sum_{k \le \frac{1}{u}} |\omega(k,u)| + \sum_{k > \frac{1}{u}} |\omega(k,u)|$$

$$= \sum\nolimits_1 + \sum\nolimits_2$$

$$\sum\nolimits_1 = \sum_{k \le \frac{1}{u}} O\left(u^\upsilon k^{\beta-1}(n+k)^\sigma\right)$$

$$- O(u^\upsilon) \sum_{k < \frac{1}{u}} k^{\beta-1}(n+k)^\sigma n\, \Omega(n)$$

$= O(u^\upsilon) O\big(n\, \Omega(n)\big) = O(1)$ Using (3.1).

Again,

$$\sum\nolimits_2 = \sum_{k > \frac{1}{u}} \omega(k,u)$$

$$= \sum_{k > \frac{1}{u}} O\left(\frac{u^\upsilon k^{\beta-1}(n+k)^\sigma\, n\, \Omega(n)}{(k+1)} \right)$$

$$\le O(u^\upsilon) \sum_{k > \frac{1}{u}} \frac{u^\upsilon k^{\beta-1}(n+k)^\sigma\, n\, \Omega(n)}{k}$$

$$\le O(u^\upsilon) \sum_{k > \frac{1}{u}} k^{\beta-1} \frac{(n+k)^\sigma k}{k} n\, \Omega(n)$$

$$= O(u^\upsilon) \sum_{k > \frac{1}{u}} k^{\beta-1}(n+k)^\sigma n\, \Omega(n)$$

$$= O(u^\upsilon) O\big(n\, \Omega(n)\big) = O(1)$$

This completes the proof of the theorem.

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