© 2016 IJSRSET | Volume 2 | Issue 1 | Print ISSN : 2395-1990 | Online ISSN : 2394-4099
Themed Section: Engineering and Technology

Oscillation Conditions of First Order Delay Differential Equations with Positive and Negative Coefficients

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ABSTRACT

In this paper, we obtain some oscillation criteria for the first order delay differential equation with $x'(t) + p(t)\, x(\tau(t)) = 0,\ t \geq t_0$. By applying these results, we also establish some integral conditions for oscillation of the higher order delay differential equations.

Keywords: Oscillation; Delay Differential equations; Linear; Positive; Negative.

I. INTRODUCTION

In this paper, oscillation criteria are established for first order delay differential Equations. Delay Differential Equations are one of the most powerful mathematical modelling tools & they arise naturally in various applications from the life sciences to engineering, physics, etc., the oscillatory behaviour of the solutions of first order linear Delay Differential Equations has been extensively studied in recent years.

The qualitative properties of oscillation of the solution to the linear delay differential equations for P, $\tau \in C\ \{[t_0, \infty), R^+\}$, $R^+ = [0, \infty)$

$x'(t) + p(t)\, x(\tau(t)) = 0,\ t \geq t_0$ -------------(1.1)

And $x'(t) + p(t)\, x(t - \tau) = 0,\ t \geq t_0$---- (1.2)

Where $p(t) \geq 0$ & $\tau(t)$ is piecewise continuous and $\tau(t)$ is a non – decreasing, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$

For (1.2) the function T defined by $T(t) = t - \tau(t),\ t \geq 0$, is increasing such that

$\lim_{t\to\infty} T(t) = \infty$

As is customary, a solution of (1.1) (or) (1.2) is said to be oscillatory if it has arbitrary large zeros.

The following assumptions will be used throughout this paper, without further mention.

Let the numbers K & L defined by

$K = \lim_{t\to\infty} \inf \int_{\tau(t)}^{t} p(s)\, ds \geq \frac{1}{e}$

$L = \lim_{t\to\infty} \sup \int_{\pi(t)}^{\tau} p(s)\, ds \geq 1$

Also $L = \lim_{t\to\infty} \sup \int_{\pi(t)}^{\tau} p(s)\, ds \geq \frac{1-k^2}{4}$

If $0 < k < \frac{1}{e}$

$L > 1 - \frac{1-k-\sqrt{1-2k-k^2}}{2}$ and

$L > \frac{l_n \lambda_1 + 1}{\lambda_1}$

Where λ_1 is the smaller root of $\lambda = e^{k\lambda}$

Set $w(t) = \frac{x[\tau(t)]}{x(t)}$ -------------------------(1.3)

Also $w(t) = \exp \int_{\pi(t)}^{\tau} p(s) w(s)\, ds$ --------------- (1.4)

$F(t) = \frac{p(t)}{\mu(t)}$ ------------------------------------ (1.5)

LEMMA: 1.1

Suppose that k>0 and Equation (1.1) has an eventually positive solution x(t) then $k < \frac{1}{e}$ and $\lambda_1 < \liminf_{t\to\infty} w(t) < \lambda_2$ where λ_1 is the smaller root and λ_2 the greater root of the equation $\lambda = e^{k\lambda}$.

PROOF:

Set $w(t) = \frac{x(\tau(t))}{x(t)}$

Let $\alpha = \lim_{t\to\infty} \inf w(t)$

From (1.4) we have sufficiently large 't'

$\alpha \geq \exp k\alpha$

Which is impossible if $k > \frac{1}{e}$

Since, this case $\lambda < e^{k\lambda}\ \forall\ \lambda$

$\Rightarrow$ (1.1) has no eventually positive solution if $k > \frac{1}{e}$

Now,

if $0 < k \leq \frac{1}{e}$ then $\lambda = e^{k\lambda}$ has roots $\lambda_1 \leq \lambda_2$

(With equality $\lambda_1 = \lambda_2 = e \Leftrightarrow k = \frac{1}{e}$)

And $\alpha \geq e^{k\alpha} \Leftrightarrow \lambda_1 \leq \alpha \leq \lambda_2$

$\therefore \lambda_1 \lim_{t\to\infty} \inf w(t) \leq \lambda_2$

LEMMA: 1.2

Let $0 < k < \frac{1}{e}$ and x(t) be an eventually positive solution of Equation (1.1). Assume that there exists $\theta > 0$ such that

$\int_{T(u)}^{\tau(t)} F(s)ds \geq \theta \int_{u}^{t} F(s)ds$ for all $\tau(t) \leq u \leq t$ ---------(A)

Then $\lim_{t\to\infty} \sup w(t) \leq \frac{2}{(1-k-\sqrt{(1-k)^2-4B}}$ ---------- (B)

Where B is given by $B = \frac{e^{\lambda\theta k} - \lambda 1\theta k - 1}{(\lambda 1\theta)^2}$ --------------(C)

And λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$

PROOF:

Let $t > t_0 \geq 1$ be large enough so that $\tau(t_1) = t$

$\delta = \int_t^{t1} F(s)ds < \int_t^{t1} P(s)ds$, where $0 < \delta$: k is arbitrary close to k

Integrating (1.1) from t to t_1, we get

$$x(t) = x(t_1) + \int_t^{t1} P(s)\ x(\tau(s))\ ds$$

And $F(s) = \frac{p(s)}{\mu(s)}$

$$x(t) = x(t_1) + \int_t^{t1} F(s)\ \mu(s)\ x(\tau(s))\ ds$$

Integrating (1.1) from $\tau(s)$ to t for s <t_1, we have

$$x[\tau(s)] = x(t) + \int_{\tau(s)}^{t} P(u)\ x(\tau(u))\ du$$

$$= x(t) + \int_{\tau(s)}^{t} F(u)\ \mu(u)\ x(\tau(u))\ du$$

Combining last two equalities, we have

$x(t) = x(t_1) + \int_t^{t1} F(s)\ \mu(s)\ \left(x(t) + \int_{\tau(s)}^{t} F(u)\ \mu(u)\ x(\tau(u))\ du \right)$ ds ----- -----(1.6)

Let $0 < \lambda < \lambda_1$, then the function

$\varphi(t) = x(t)\ e^{\lambda \int_{t0}^{t1} F(s)\ ds}$ ------------------(1.7) is decreasing for large $t > t_0$

Since x(t) also decreasing

From lemma (1.1) $\frac{x(\tau(t))}{x(t)} > \tau$

Since $\mu(t) \geq 1$ for $t \geq t_0 \geq 1$

Then $\frac{\mu(t)x(\tau(t))}{x(t)} > \lambda$ for all sufficiently large 't'

$0 = x^1(t) + F(t)\ \mu(t)\ x(\tau(t)) \geq x^1(t) + \lambda F(t)\ x(t)$

$\Rightarrow \varphi(t) \leq 0$ for sufficiently large t substituting into (1.6), we get for sufficiently large 't' the inequality

$x(t) \geq x(t_1) + \delta\ x(t) + \int_t^{t1} F(s) \left(\int_{\tau(s)}^{t} \mu(u)\ \varphi(\tau(u)) e^{-\lambda \int_{t0}^{\tau(u)} F(\xi)d\xi} du\right)$ds

$\geq x(t_1) + \delta\ (x(t) + \mu(t)\ \varphi(\tau(t))\ \int_t^{t1} F(s) \left(\int_{\tau(s)}^{t} F(u)\ e^{-\lambda \int_{t0}^{\tau(u)} F(\xi)d\xi} du\right)$ds

$= x(t_1) + \delta\ (x(t) + \mu(t)\ \varphi(\tau(t))\ e^{-\lambda \int_{t0}^{\tau(t)} F(s)ds} \int_t^{t1} F(s) \left(\int_{\tau(s)}^{t} F(u)\ e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi)d\xi} du\right)$ds

From (1.7)

$x(t) \geq x(t_1) + \delta\ (x(t) + \mu(t)\ x(\tau(t))\ \int_t^{t1} F(s) \left[\int_{\tau(s)}^{\tau} F(u)\ e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi)d\xi} du\right]$ds ---- --------- (1.8)

In view of (A) we obtain

$\int_{\tau(s)}^{t} F(u)\ e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi)d\xi} du \geq \int_{\tau(s)}^{t} F(u)\ e^{\lambda\theta \int_{u}^{t} F(\xi)d\xi} du = \frac{1}{\lambda\theta}\left(e^{\lambda\theta \int_{\tau(s)}^{t} F(\xi)d\xi} - 1\right)$

$\therefore \int_t^{t1} F(s) \left(\int_{\tau(s)}^{t} F(u)\ e^{\lambda \int_{\tau(u)}^{\tau(t)} F(\xi)d\xi} du\right) ds \geq$

$-\frac{\delta}{\lambda\theta} + \frac{1}{\lambda\theta} \int_t^{t1} F(s)\ e^{\lambda\theta \int_{\tau(s)}^{t} F(\xi)d\xi}$

$= \frac{-\delta}{\lambda\theta} + \frac{1}{\lambda\theta} \int_{t1}^{t} F(s)\ e^{\lambda\theta \int_{\tau(s)}^{t} F(\xi)d\xi - \lambda\theta \int_{t}^{s} F(\xi)d\xi}$ds

$\geq \frac{-\delta}{\lambda\theta} + \frac{1}{\lambda\theta}\ e^{\lambda\theta\delta} \int_t^{t1} F(s)\ e^{-\lambda\theta \int_t^{s} F(\xi)d\xi}$ds

$= \frac{-\delta}{\lambda\theta} + \frac{e^{\lambda\theta\delta}}{(\lambda\theta)^2}\left(1 - e^{-\lambda\theta \int_t^{t1} F(\xi)d\xi}\right)$

$= \frac{-\delta}{\lambda\theta} + \frac{e^{\lambda\theta\delta}}{(\lambda\theta)^2}\left(1 - e^{-\lambda\theta\delta}\right)$

$= \frac{-\delta}{\lambda\theta} + \frac{e^{\lambda\theta\delta}}{(\lambda\theta)^2}\left(e^{-\lambda\theta\delta} - 1\right)$

From (1.8) yields

x(t) > x(t$_1$) + δx(t) + B* μ(t) x($\tau(t)$) --------------------(1.9)

Where $B^* = \frac{e^{\lambda\theta\delta} - \lambda\theta\delta - 1}{(\lambda\theta)^2}$

From (1.9), we have x(t) $\geq$ d_1 $\mu(t)$ x(τ(t))

Where $d_1 = \frac{B^*}{1-\delta}$

Observe that x(t$_1$) $\geq$ d_1 $\mu(t1)$ x(τ(t1)) $\geq$ d_1 x(t)

Since $\mu(t) \geq 1$ for t $\geq$ $t_0 \geq 1$

$\therefore$ (1.9) $\Rightarrow$ x(t) $\geq$ d_2 $\mu(t)$ x(τ(t))

Where $d_2 = \frac{B^*}{1-d1-\delta}$

Using I derivative procedure, then

x(t) $\geq$ d_{n+1} $\mu_{(t)}$ x(τ(t))

Where $d_{n+1} = \frac{B^*}{1-dn-\delta}$, n=1, 2, 3, ...

It is easy to see that the sequence {d_n} is strictly increasing and bounced

$\therefore$ $\lim_{t\to\infty}$ d_n = d exists and Satisfies $d^2 - (1-\delta)d + B^* = 0$

$\because$ {d_n} is strictly increasing it follows that

$$d = \frac{1-\delta-\sqrt{(1-\delta)^2 - 4B^*}}{2}$$

Observe that for large t one has

$$\frac{x(t)}{\mu(t)\ x(\tau(t))} \geq \frac{1-\delta-\sqrt{(1-\delta)^2-4B^*}}{2}$$

And since 0 < δ < k is arbitrarily close to k, by letting $\lambda \to \lambda_1$, it leads to (B)

The proof is complete.

REMARK:

Assume that $\tau(t)$ is continuously differentiable and that there exists $\theta > 0$

F(τ(t)) τ^1(t) $\geq$ θ F(t) ------------------------(1.10)
eventually for all t.

Then it is easy to see that (1.10) implies (A)

The function

V(u) = $\int_{t(u)}^{\tau(t)} F(s)$ds - $\theta\int_u^t F(s)$ds $\tau(t) \leq u \leq t$

satisfies the conditions v(t) = 0

And $v^1(u) = -F(\tau(u))\,\tau^1(u) + \theta F(u) \leq 0$

If F(t) > 0 eventually for all 't' and

$\lim_{t\to\infty}$ inf $\frac{F(\tau(t))\ \tau 1(t)}{F(t)} = \theta_0 > 0$

Then θ can be any number satisfying $0 < \theta < \theta_0$

LEMMA: 1.3

Assume that (1.1) has an eventually positive solution x(t). Set

B(t) = max $\left\{\frac{x(s)}{x\tau(s)};\ \tau(t) \leq s \leq t\right\}$

Then $\lim_{t\to\infty}$ inf B(t) > $\frac{1}{\lambda^2}$ --------------------------(C)

PROOF:

Assume for the sake of contradiction that (C) is not true. Then there exist an increasing sequence {t_n} with $t_n \to \infty$ as n→∞ such that

$\lim_{n\to\infty}$ B(t_n) = $\lim_{n\to\infty}$ inf B(t) = $\mu < \frac{1}{\lambda^2}$

For a given $\lambda \in (\mu, \frac{1}{\lambda^2})$, there exists an integer N>0 such that

B(t_n) <λ, n$\geq$ N------------------------------------ (1.11)

Since -λ l_n λ < $\frac{\ln \lambda 2}{\lambda 2}$ = k,

It follows from the definition of k that there exists an integer N_1 > N such that

$\int_{\tau(t)}^{\tau} p(s)ds$ > - λ $l_n\lambda$, t $\geq$ t_{N1} ----------------------- (1.12)

Next we prove that $\frac{x(t)}{x(\tau(t))} < \lambda$, t $\geq$ t_{N1}---------------- (1.13)

In fact, if (1.13) is not true, then by (1.11) there exists an integer $n_1 \geq N_1$ and T with $t_{n1} \leq T < t_{n1} + 1$ such that

$\frac{x(t)}{x(\tau(t))} < \lambda$ for t ∈ [τ(t_{n1}), T) &

$\frac{x(t)}{x(\tau(t))} = \lambda$

By (1.1) we have

$\int_{\tau(t)}^{T\tau} p(s)ds = \int_{\tau(t)}^{T\tau} \frac{x^1(s)}{x(\tau(s))}$ds $\leq$ l_n $\frac{x(\tau(t))}{x(t)}$. B(T) < -λ l_n λ

Which contradicts (1.12) and so (1.13) holds.

We have $\lim_{t\to\infty}$ inf $\frac{x(\tau(t))}{x(t)} = \lim_{t\to\infty}$ inf w(t) $\geq \frac{1}{\lambda} > \lambda_2$

Which contradicts (lemma 1.1)

The proof is complete

LEMMA: 1.4

If $\lim_{t\to\infty}$ sup $\int_t^{t+\tau i} Pi(s)x\,(s-\tau i)\,ds \leq 0$, for some i, and x(t) eventually positive solution of $x'(t) + \sum_{i=1}^{n} pi(t)x(t-\tau i) = 0$, then for the same i,

$\lim_{t\to\infty}$ inf $\frac{x(t-\tau i))}{x(t)} < \infty$ ------------------------------- (1.14)

PROOF:

There exist a constant d > 0 and a sequence $\{t_k\}$ such that $t_k \to \infty$ as $k\to\infty$ and

$\int_{tk}^{tk+\xi i} Pi(s)ds \geq d$, k = 1,2 ...

For all $\xi_k \in (t_k, t_k+\tau i)$ for every k such that

$\int_{tk}^{\xi k} Pi(s)ds \geq d/2$ and -------------------(1.15)

$\int_{\xi k}^{tk+ti} Pi(s)ds \geq d/2$

Then $x'(t) + P_i(t)\,x(t-\tau i) \leq 0$ -----------(1.16)

Eventually

Integrating (1.16) with $[t_k, \xi k]$ & $[\xi k, t_k + \tau i]$

(1.16) $\Rightarrow \int_{tk}^{\xi k} x^1(t)dt + \int_{tk}^{\xi k} Pi(t)x(t-\tau i)\,dt \leq 0$

$\Rightarrow x(t)\,]_{tk}^{\xi k} + \int_{tk}^{\xi k} Pi(s)\,x(t-\tau i)\,ds \leq 0$

$\Rightarrow x(\xi_k) - x(t_k) + \int_{tk}^{\xi k} Pi(s)\,x(s-\tau i)\,ds \leq 0$ ---------- (1.17)

And

$x(t_k + \tau i) - x(\xi_k) + \int_{\xi k}^{tk+\tau i} Pi(s)\,x(s-\tau i)\,ds \leq 0$ ---- (1.18)

By omitting first term in (1.17) & (1.18) by using the decreasing nature of x(t) and (1.15),

We find, (1.17) $\Rightarrow$

$-x(t_k) + \int_{tk}^{\xi k} Pi(s)\,x(s-\tau i)\,ds \leq 0$

$\Rightarrow -x(t_k) + \frac{d}{2}x(\xi_{k-}\tau i) \leq 0$

(1.18) $\Rightarrow -x(\xi_k) + \frac{d}{2}x(t_k) \leq 0$

(OR)

$(\frac{x(\xi k-\tau i)}{x(\xi k)} < \left(\frac{d}{2}\right)^2$

This completes the proof

THEOREM: 1.1

Consider the Differential Equation (1.1) and let L<1, $0<k<\frac{1}{e}$ and there exists θ>0 such that (A) is satisfied.

Assume that $l > \frac{l_n\lambda_1+1}{\lambda_1} - \frac{1-k-\sqrt{(1-k)^2-4B}}{2}$--------------- (D)

Where λ_1 is the smaller root of the equation $\lambda = e^{k\lambda}$ and B is given by (C). Then all solutions of (1.1) oscillate.

PROOF:

Assume, for the sake of contradiction, that x(t) is eventually positive solution of (1.1)

Let σ be any number $(\frac{1}{\lambda_1}, 1)$

From Lemma (1.1), there is a $T_1 > t_0$ such that

$\frac{x(\tau(t))}{x(t)} > \sigma\lambda_1$, $t \geq T_1$ ----------------------------------- (1.19)

$\frac{x(t)}{x(\tau(t))} > \sigma M$, $t \geq T_1$ ----------------------------------- (1.20)

Where $M = \lim_{t\to\infty}$ inf $\frac{x(t)}{x(\tau(t))}$

Now let $t > T_1$.

Since the function $g(s) = \frac{x(\tau(t))}{\tau(s)}$ is continuous,

$g(\tau(t)) = 1 < \sigma\lambda_1$

$g(t) > \sigma\lambda_1$

There is a $t^*(t) \in (\tau(t), t)$ such that

$\frac{x(\tau(t))}{x(t^*(t))} = \sigma\lambda_1$

Dividing (1.1) by x(t)

$\frac{x^1(t)}{x(t)} + \frac{P(t)x(\tau(t))}{x(t)} = 0$

Integrating from $\tau(t)$ to $t^*(t)$ & use (1.19)

$\Rightarrow \int_{\tau(t)}^{t^*(t)} \frac{x^1(s)}{x(s)}ds + \int_{\tau(t)}^{t^*(t)} \frac{P(s)x(\tau(s))}{x(s)}\,ds = 0$

$\Rightarrow \int_{\tau(t)}^{t^*(t)} p(s)(\sigma\lambda 1)\,ds \leq -\int_{\tau(t)}^{t^*(t)} p(s)\frac{x^1(s)}{x(s)}$

$\Rightarrow (\sigma\lambda_1)\int_{\tau(t)}^{t^*(t)} p(s)ds \leq -\int_{\tau(t)}^{t^*(t)} \frac{x^1(s)}{x(s)}\,ds$

$\Rightarrow \int_{\tau(t)}^{t^*(t)} p(s)ds \leq \left(-\frac{-1}{\sigma\lambda 1}\right)\int_{\tau(t)}^{t^*(t)} \frac{x^1(s)}{x(s)}\,ds$

$= \left(\frac{1}{\sigma\lambda 1}\right)\int_{\tau(t)}^{t^*(t)} \frac{x^1(s)}{x(s)}\,ds$

$= \left(\frac{1}{\sigma\lambda 1}\right)\ln[x(s)]_{t^*(t)}^{\tau(t)}$

$= \left(\frac{1}{\sigma\lambda 1}\right)\ln x(\tau(t)) - \ln x(t^*(t))$

$= \left(\frac{1}{\sigma\lambda 1}\right)\ln\left(\frac{x(\tau(t))}{x(t^*(t))}\right)$

$= \left(\frac{1}{\sigma \lambda 1}\right) \ln (\sigma \lambda_1)$

$\therefore \int_{\tau(t)}^{t^*(t)} p(s)ds = \frac{\ln (\sigma \lambda_1)}{(\sigma \lambda_1)}$ ---------------------------- (1.21)

Integrating (1.1) over [t^* (t), t] and using (1.20) and $x(\tau(s)) \geq x(\tau(s))$ if s $\leq$ t yields

$\int_{t^*(t)}^{t} p(s)ds < \left(\frac{x(t^*(t))}{x(\tau(t))}\right) - \left(\frac{x(t)}{x(\tau(t))}\right)$

$= \frac{1}{\sigma \lambda 1} - \left(\frac{x(t)}{x(\tau(t))}\right)$

$= \frac{1}{\sigma \lambda 1} - \sigma$---(1.22)

(1.21) + (1.22) $\Rightarrow$

$\int_{\tau(t)}^{t^*(t)} p(s)ds + \int_{t^*(t)}^{t} p(s)ds \leq \frac{\ln (\sigma \lambda_1)}{(\sigma \lambda_1)} + \frac{1}{\sigma \lambda 1} - \sigma M$

$\Rightarrow \int_{\tau(t)}^{t} p(s)ds \leq \frac{\ln (\sigma \lambda_1)+1}{(\sigma \lambda_1)} - \sigma M$

Letting

$t \rightarrow \infty \int_{\tau(t)}^{t} p(s)ds \leq \frac{\ln (\sigma \lambda_1)+1}{(\sigma \lambda_1)} - \sigma M$

$l \leq \frac{\ln (\sigma \lambda_1)+1}{(\sigma \lambda_1)} - \sigma M$

Letting $\sigma \rightarrow 1$

$l \leq \frac{\ln (\sigma \lambda_1)+1}{(\lambda_1)} - M$

The last inequality, in view of Lemma (1.2) contradicts (D) Hence Proved.

THEOREM: (1.2)

Suppose that $\int_{t}^{t+\tau} p(s)\, ds > 0$ for t $\geq$ t_0 for some t_0 > 0 and

$\int_{t_0}^{\infty} p(t) \ln \left(e \int_{t}^{t+\tau} p(s)\, ds\right)dt = \infty$ --- (E)

Then every solution of x′(t) + p(t) x(t-τ) = 0 oscillates.

PROOF:

Assume the contrary.

Then we have an eventually positive solution x(t) of x′(t) + p(t) x(t-τ) = 0

So, x(t) is eventually monotonically decreasing

Let $\lambda = \frac{-x1(t)}{x(t)}$

Clearly for large'ť', function λ(t) is non-negative and continuous and

$x(t) = x(t_1) \exp\left[-\int_{t_1}^{t} \lambda(s)ds\right]$ where $x(t_1) > 0$ for some $t_1 > t_0$.

Also λ(t) satisfies the generated characteristic equation

$\lambda(t) = p(t) \exp\left[\int_{t-\tau}^{t} \lambda(s)ds\right]$ -------------------------- (1.23)

We can easily show that

$e^{rx} \geq x + \frac{\ln(er)}{r}$ for r>0 ---------------------- (1.24)

Thus (1.23) becomes

$\lambda(t) = p(t) \exp\left[A(t)\frac{1}{A(t)}\int_{t-\tau}^{t} \lambda(s)ds\right]$

Using (1.24)

$\lambda(t) \geq p(t)\left[\frac{1}{A(t)}\int_{t-\tau}^{t} \lambda(s)ds + \frac{\ln(e\,A(t))}{A(t)}\right]$

Where $A(t) = \int_{t}^{t+\tau} p(s)\, ds$

Then

$A(t)\, \tau(t) \geq p(t) \int_{t-\tau}^{t} \lambda(s)ds + p(t) \ln(e\, A(t))$

$\tau(t) \int_{t}^{t+\tau} p(s)\, ds \geq p(t) \int_{t-\tau}^{t} \lambda(s)ds + p(t) \ln\left[e \int_{t-\tau}^{t} \lambda(s)ds\right]$

$\tau(t) \int_{t}^{t+\tau} p(s)\, ds - p(t) \int_{t-\tau}^{t} \lambda(s)ds \geq p(t) \ln\left[e \int_{t-\tau}^{t} \lambda(s)ds\right]$

Then for N > T & integrating

$\therefore \int_{T}^{N} \tau(t) \int_{t}^{t+\tau} p(s)\, ds - \int_{T}^{N} P(t) \int_{t-\tau}^{t} \lambda(s)dt \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

Consider

$\int_{T}^{N} P(t) \int_{t-\tau}^{t} \lambda(s)dsdt \geq \int_{T}^{N-T}\left(\int_{S}^{s+\tau} p(t)\tau(s)dt\right)ds$

(By interchange of order of integration)

$= \int_{T}^{N-\tau} \lambda(s) \int_{S}^{S+\tau} P(t)dt\, ds$

$= \int_{T}^{N-\tau} \lambda(t) \int_{t}^{t+\tau} P(s)ds\, dt$--------------------------- (1.25)

Using (1.25) in (1.24) $\Rightarrow$

$\int_{T}^{N} \tau(t) \int_{t}^{t+\tau} P(s)ds\, dt - \int_{T}^{N-\tau} \lambda(t) \int_{t}^{t+\tau} P(s)ds\, dt \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

$\Rightarrow \int_{N}^{N-\tau} \lambda(t) \int_{t}^{t+\tau} P(s)\, ds\, dt \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

We have $\int_{t}^{t+\tau_i} P_i(s)ds \leq 1$, i=1, 2 ...n

$\Rightarrow \int_{N-\tau}^{N} \lambda(t) \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

$\Rightarrow \int_{N-\tau}^{N} \frac{-x^1(t)}{x(t)} \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

$\Rightarrow -[\ln x(t)]_{N-\tau}^{N} \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

$\Rightarrow \ln x(N-\tau) - \ln x(N) \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$

$\Rightarrow \ln\frac{x(N-\tau)}{x(N)} \geq \int_{T}^{N} P(t) \ln\left[e \int_{t}^{t+\tau} P(s)ds\right]dt$-------- (1.26)

Given $\int_T^N P(t) \ln \left[e \int_t^{t+\tau} P(s)ds\right] dt = \infty$

by (E)

$\therefore$ (1.25) $\Rightarrow$

$\lim_{t\to\infty} \frac{x(t-\tau)}{x(t)} = \infty$ ---------------------------------- (1.27)

Now (E) implies that there exist a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$\int_{t_n}^{t_n+\tau} p(s)ds > \frac{1}{e}$ for all n

Hence by lemma (1.4), we obtain

$$\lim_{t\to\infty} inf \frac{x(t-\tau)}{x(t)} < \infty$$

This contradicts (1.26) & completes the proof.

THEOREM: 1.3

Assume that $0 < \propto < \frac{1}{e}$ and

$\lim_{t\to\infty} sup \left\{ \begin{matrix} min \\ \tau(t) \le s \le t \end{matrix} \int_{\tau(s)}^{s} p(\xi) d\xi \right\} > \frac{1+l_n ⋋_1}{⋋_1} - \frac{1}{⋋_2}$ --------------------- (F)

Then all solutions of (1.1) oscillate.

PROOF:

Assume, for the sake of contradiction, that (1.1) has an eventually positive solution x(t).

For given $\theta \in (0.1)$ by lemma (1.3)

$\int_{\tau(t)}^{t} p(s)\, ds \ \ge \ \theta \ \propto$ & $\frac{x(\tau(t))}{x(t)} \ge \theta ⋋_1$

For all sufficiently large t, and consequently for $\tau(t) \le s \le t$

$\frac{x(\tau(s))}{x(s)} = \exp \int_{\tau(s)}^{\tau(t)} p(\xi) \frac{x(\tau(\xi))}{x(\xi)} d\xi$

$\frac{x(\tau(s))}{x(s)} \ge \exp\left(\theta ⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi\right) \ge e^{(\theta-1)⋋_1} \exp((1-\theta) ⋋_1 + \theta ⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi)$

$\ge e^{(\theta-1)⋋_1} \exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi)$ -- (1.28)

Since $\int_{\tau(s)}^{s} p(\xi) d\xi \le 1$

Integrating (1.1) from $\tau(t)$ to t & using (1.28)

$\int_{\tau(t)}^{t} x^1(s)ds + \int_{\tau(t)}^{t} p(s)x(\tau(s))ds = 0$

$x(s)]_{\tau(t)}^{t} + \int_{\tau(t)}^{t} p(s)x(\tau(s))ds = 0$

$x(t) - x(\tau(t)) + \int_{\tau(t)}^{t} p(s)x(\tau(s))ds = 0$

$x(\tau(t)) - x(t) = \int_{\tau(t)}^{t} p(s)x(\tau(s))ds = 0$

$\ge e^{(\theta-1)⋋_1} x(\tau(t)) \int_{\tau(t)}^{t} p(s) \exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi)$

$x(\tau(t)) \ge x(t) + e^{(\theta-1)⋋_1} x(\tau(t)) \int_{\tau(t)}^{t} p(s) \exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi)$

$1 > \frac{x(t)}{x(\tau(t))} + e^{(\theta-1)⋋_1} x(\tau(t)) \int_{\tau(t)}^{t} p(s) \exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi) ds$ ------ (1.29)

Let t be large enough so that

$\int_{\tau(t)}^{t} p(s)ds \ge \theta\alpha$

There exists $t^* \in [\tau(t), t]$ such that $\int_{\tau(t)}^{t^*} p(s) = \theta\alpha$

Thus

$\int_{\tau(t)}^{t} p(s) \exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi) ds \ge$

$\int_{\tau(t)}^{t} p(s)ds + \int_{\tau(t)}^{t^*} p(s) [\exp(⋋_1 \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi - 1)] ds$

$= \int_{\tau(t)}^{t} p(s) ds + \int_{\tau(t)}^{t^*} p(s) \left[\left\{⋋_1 (\int_{\tau(s)}^{s} p(\xi) d\xi - \int_{\tau(s)}^{\tau(t)} p(\xi) d\xi)\right\} - 1\right] ds$

$\ge \int_{\tau(t)}^{t} p(s)ds + e^{\theta\propto⋋_1} \int_{\tau(t)}^{t^*} p(s) \exp(⋋_1 \int_{\tau(s)}^{s} p(\xi) d\xi) ds - \theta\propto$

$= \int_{\tau(t)}^{t} p(s)ds + \frac{e^{\theta\propto⋋_1} - (1+\theta\propto⋋_1)}{⋋_1}$

Substituting this into (1.28) we have

$1 \ge \frac{x(t)}{x(\tau(t))} + e^{(\theta-1)⋋_1} \left[\int_{\tau(t)}^{t} p(s)\, ds + \frac{e^{\theta⋋_1} - (1+\theta⋋_1\propto)}{⋋_1}\right]$

$e^{(\theta-1)⋋_1} - \frac{e^{\theta\propto⋋_1} - (1+\theta\propto⋋_1)}{⋋_1} \ge e^{(\theta-1)⋋_1} B(t) + \min \int_{\tau(s)}^{s} p(\xi) d\xi$

Taking superior limit as $t\to\infty$ & using lemma (1.4)

$e^{(\theta-1)⋋_1} - \frac{e^{\propto\theta⋋_1} - (1+\theta\propto⋋_1)}{⋋_1} \ge \lim_{t\to\infty} sup(e^{(\theta-1)⋋_1} B(t) + \min \int_{\tau(s)}^{s} p(\xi) d\xi)$

$\ge \frac{1}{⋋_2} e^{(\theta-1)⋋_1} + \lim_{t\to\infty} sup \left\{\min_{\tau(t)\le s\le t} \int_{\tau(s)}^{s} p(\xi) d\xi\right\}$

Since $0 < \theta < 1$is arbitrarily close to 1

We let $\theta \to 1$

Then $\lim_{t\to\infty} sup\left\{\min_{\tau(t)\le s\le t}\int_{\tau(s)}^{s} p(\xi)d\xi\right\}$ < $\frac{1-e^{\alpha\lambda_1}-(1+\theta\alpha\lambda_1)}{\lambda_1}$ - $\frac{1}{\lambda_2}$

= $\frac{1+\ln\lambda_1}{\lambda_1}$ - $\frac{1}{\lambda_2}$

Which contradicts (F) and so the proof is complete.

Hence Proved.

II. ON OSCILLATION PROPERTIES OF DELAY DIFFERENTIAL EQUATIONS WITH A POSITIVE AND A NEGATIVE TERM

Delay differential equations having forms

x'(t)+P (t) x (t-τ)-Q (t) x (t-σ)=0, t≥ t_0------------ (2.1)

And [x(t)-R(t)x(t-ρ)]' + P(t)x(t-τ)-Q(t)x(t-σ)=0, t≥ t_0 where P, Q, R ∈ $C([t_0,\infty), R^+)$ and $\tau, \sigma, \rho \ge 0$

The following assumptions will be used this chapter without further mention

Eq.(2.1) is oscillatory when

$\tau \ge \sigma \ge 0$

p>q>0 ------------------------- (2.2)

q(τ -σ)≤ 1

(p-q) >(1/e)(1-q($\tau-\sigma$))

Under conditions

$\lim_{t\to\infty} inf \int_{t-\tau}^{t} \bar{P}$(s) ds > (1/e)

$\lim_{t\to\infty} sup \int_{t-\tau}^{t} \bar{P}$(s) ds > 1

LEMMA: 2.1

Assume that x (t) is eventually positive solution of (2.1) holds. Then for n∈ N eventually positive z(t) in

z (t) = x (t) - $\int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)$ ds

satisfies

z'(t) + $\bar{P}$(t) $\sum_{i=0}^{n} Q_i$ (t-τ) z (t-τ) ≤ 0 ----------------- (2.3) eventually.

PROOF:

Assume that x (t) is eventually positive solution of (2.1) Then there exists a $t_1 \ge t_0$ such that x (t)>0 for $t \ge t_1$

Set t_2 =max {$t_1+\tau$, t}

Since

z (t) = x (t) - $\int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)$ ds---------------- (2.4)

Satisfies z'(t) ≤ 0 , z (t) > 0

We have 0< z (t) ≤ x (t), $t \ge t_2$ ------------------ (2.5)

From (2.4)

x (t) = z (t) - $\int_{t-\tau+\sigma}^{t} Q(s)x(s-\sigma)$ ds $t \ge t_2$

z(t)+ $\int_{t-\tau+\sigma}^{t} Q(s_1)[z(s_1-\sigma)$ + $\int_{s_1-\tau+\sigma}^{s_1} Q(s_2)x(s_2-\sigma)ds_2$]d$s_1$=x(t),

$\ge t_2 + \sigma$

Since z'(t) ≤ 0 we have

x (t) ≥ z(t)+z($t-\sigma$) $\int_{t-\tau+\sigma}^{t} Q(s)$ ds $+\int_{t-\tau+\sigma}^{t} Q(s_1)\int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma)x(s_2-2\sigma)ds_2ds_1$

≥ z(t) [1+ $\int_{t-\tau+\sigma}^{t} Q(s)$ ds] $+\int_{t-\tau+\sigma}^{t} Q(s_1)\int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma)x(s_2-2\sigma)ds_2ds_1$

=z(t)[$Q_0(t)+Q_1$ (t)]+ $\int_{t-\tau+\sigma}^{t} Q(s_1)\int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma)x(s_2-2\sigma)ds_2ds_1$

x(t) =z(t) $\sum_{i=0}^{1} Q_i$ (t)+ $\int_{t-\tau+\sigma}^{t} Q(s_1)\int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma)x(s_2-2\sigma)ds_2ds_1$ for $t \ge t_2$+ σ

Repeating the above procedure for n-times, we have

z(t) $\sum_{i=0}^{1} Q_i$(t)+ $\int_{t-\tau+\sigma}^{t} Q(s_1)$.........$\int_{s_{n-\tau+\sigma}}^{s_n} Q(s_{n+1}$ –nσ) x(s_{n+1}-(n+1)σ)ds_{n+1}...... ds_1 ≤ x(t)

(or)

z(t) $\sum_{i=0}^{1} Q_i$(t) ≤ x(t) for $t \ge t_2$+n σ -----------------(2.6)

Since z'(t) +$\overline{P(t)}$ x (t- σ) = 0

We have, z'(t) +$\overline{P(t)}$ $\sum_{i=0}^{n} Q_i$(t- σ) z(t-τ) ≤ 0 , $t \ge t_2$+n σ+τ by considering (2.5) and (2.6)

Hence proved

LEMMA: 2.2

Assume that all conditions of lemma (2.1) are held. Furthermore, assume that there exists an nϵ N such that α(n) > 1/e ---------- (2.7)

(Or) α(n) ≤ 1/e, β(n) > 1- $\frac{1-\alpha(n)-\sqrt{1-2\alpha(n)-\alpha^{2}(n)}}{2}$----

(2.8) holds. Then every solution of (3) is oscillatory.

PROOF:

Assume for contrary that x(t) is an eventually positive solution of (2.1)

Then in the view of (2.7) and (2.8) z (t) in x (t) - $\int_{t-\tau+\sigma}^{t} Q(s)$x(s-$\sigma$) ds cannot be an eventually positive solution of (2.2)

This contradiction completes the proof.

LEMMA: 2.3

Assume that x (t) is an eventually positive solution of (2.1) and 0≤ R(t)≤ 1 hold. Then for nϵ N ,eventually positive z(t) in z(t)=x(t)-R(t)x(t- ρ) is a solution of the following inequality z'(t)+ p(t)$\sum_{i=0}^{n} R_i$ (t-τ) z(t- τ)≤0 --------------(2.9)

PROOF:

Assume that x (t) is an eventually positive solution of (2.1)

Then there exists $t_1 \leq t_0$ ∋ x (t) > 0 for t ≥ $t_1 - \tau$

z (t)= x(t) -R(t) x(t- ρ) satisfies z'(t) ≤ 0, 0 < z(t)

We have

0 < z (t)≤ x (t- ρ) = x (t), t ≥ t_1 --------------- (2.10)

z (t) +R(t) x(t- ρ) = x(t), t ≥ t_1

We have

z (t)+R(t)[z(t- ρ)+R(t- ρ) x(t-2ρ)]=x(t), t ≥ t_1 +ρ

and considering non-decreasing behaviour of z(t).

z(t)[1+R(t)]+R(t)R(t- ρ)x(t-2ρ) ≤ x(t), t ≥ t_1 +ρ

(i.e.) z (t) $\sum_{i=0}^{1} R_i$ (t) + R_2 (t) x (t-2ρ) ≤ x (t), t ≥ t_1 +ρ

Assume

z (t) $\sum_{i=0}^{n} R_i$(t) +R_{n+1} (t) x (t-(n+1)ρ) ≤ x (t), t ≥ t_1 +$n\rho$

(Or)

z (t) $\sum_{i=0}^{n} R_i$(t) ≤ x (t), t ≥ t_1 +$n\rho$ -------------------(2.11) for nϵ N

Since z' (t) +P (t) x (t-τ) =0

We have

z'(t) +P (t) $\sum_{i=0}^{n} R_i$(t-τ) z (t- τ)≤0, t ≥ t_1 +$n\rho$+τ, nϵ N from (2.10)

Hence Proved

THEOREM: 2.1

Assume that conditions of lemma (2.1) are satisfied and Q(t) is a non-increasing function then if there exists nϵ N such that

$$\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \bar{P}(s)\sum_{i=0}^{n}[Q(s-\tau)(t-\tau)]^i \, ds > 1/e.$$

Then every solution of (2.1) is oscillatory.

PROOF:

Consider $Q_i(t) = \begin{cases} 1 & , i=0 \\ \int_{t-\tau+s}^{t} Q(s) Q_{i-1}(s-\sigma) ds & , \sigma, i \epsilon N \end{cases}$

Q_0(t)=1

$$Q_1(t) = \int_{t-\tau+\sigma}^{t} Q(s) ds$$

$$\geq Q(t)(\tau-\sigma)$$

$$Q_2(t) = \int_{t-\tau+\sigma}^{t} Q(s) Q_1(s-\sigma) ds$$

$$\geq (t-\sigma) \int_{t-\tau+\sigma}^{t} Q(s) Q(s-\sigma) ds$$

$$\geq [Q(t)(\tau-\sigma)]^2$$

$Q_i(t) \geq [Q(t)(\tau-\sigma)]^i$, i$\epsilon N$ for sufficiency large t

Then

α (n) ≥ $\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \bar{P}(s)\sum_{i=0}^{n}[Q(s-\tau)(t-\tau)]^i$ ds

But α(n) > 1/e (by (2.7))

(i.e.) ($\lim_{t\to\infty} \inf \int_{t-\tau}^{t} \bar{P}(s)\sum_{i=0}^{n}[Q(s-\tau)(t-\tau)]^i$ ds > 1/e

Hence Proved

THEOREM: 2.2

Assume that (2.2) holds. Then every solution of (2.1) is oscillatory.

PROOF:

First of all, we calculate $\boldsymbol{Q_i}$ (t) functions

Clearly $\boldsymbol{Q_0}$(t) =1

$\boldsymbol{Q_1}$ (t) = $\int_{t-\tau+\sigma}^{t} qQ_0$ (s-$\boldsymbol{\sigma}$) ds =q $(\boldsymbol{\tau-\sigma})$

$\boldsymbol{Q_2}$ (t) = $\int_{t-\tau+\sigma}^{t} qQ_1$ (s-$\boldsymbol{\sigma}$) ds =$[q(\tau-\sigma)]^2$

∴ $\boldsymbol{Q_i}$ (t) ≥ $[\mathbf{q}(\tau-\sigma)]^i$, iϵN

<u>CASE: 1</u>

$$\mathbf{q}(\tau-\sigma) < 1$$

In this case,

$$\alpha(\infty) = \lim_{t\to\infty} inf \int_{t-\tau}^{t}(p-q)\sum_{i=0}^{n}[q(\mathbf{t}-\tau)]^i \text{ ds}$$

$$= \tau \text{ (p-q) } [\frac{1}{1-q(\tau-\sigma)}]$$

$\alpha(\infty)$ >1/e (by 2.2)

And all solutions of (2.1) are oscillatory by theorem 2.1

<u>CASE: 2</u>

$$\mathbf{q}(t-\sigma) = 1$$

In this case, $\alpha(\infty) = \infty > 1/e$

∴ Every solution of (2.1) is oscillatory

Hence proved.

THEOREM: 2.3

Assume that 0≤r (t) ≤1 and 0≤p,ρ,τ . If τ p >1/e (1-r) --------------- (2.12)

holds, Then every solution of [x(t)-r x(t- ρ)]'+p x(t-τ)=0 is oscillatory.

PROOF:

We need to calculate $\boldsymbol{R_i}$ (t) functions

$\boldsymbol{R_i}$ (t) = r^i , t ≥ t_0 + i ρ, i ϵN

CASE: 1 r < 1

Thus

$$\alpha(\infty) = \lim_{t\to\infty} inf \int_{t-\tau}^{t} p\sum_{i=0}^{\infty}[r]^i \text{ ds}$$

$$\alpha(\infty) = \frac{\tau\rho}{1-r}$$

$$= \frac{\tau\rho}{1-r} > 1/e \text{ (by 2.12)}$$

$$\therefore \alpha(\infty) > 1/e$$

Every solution of (2.12) is oscillatory

CASE: 2 r=1

Thus

$$\alpha(\infty) = \infty > 1/e$$

∴ Eq. (2.12) is oscillatory

Hence proved

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