© 2016 IJSRSET | Volume 2 | Issue 2 | Print ISSN : 2395-1990 | Online ISSN : 2394-4099
Themed Section: Engineering and Technology

A Common Fixed Point Theorem for Weakly Compatible Mapping in Fuzzy Metric Space

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ABSTRACT

The present paper we establish a common fixed point theorem for weakly compatible pair of four maps in a fuzzy metric space.

Keywords: T-Norm, Common Fixed Point, Fuzzy Metric Space, Weakly Compatible Maps.

I. INTRODUCTION

In 1965, Zadeh [1] introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life .However, when the uncertain is due to fuzziness rather than randomness; it seems that the concept of a fuzzy metric space is more suitable. we can divide them into following two groups: The first group involves those results in which a fuzzy metric on a set X is treated as a map where X represent the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distance are set up between fuzzy objects. on the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy . Kramosil and Michalek [2] have introduced the concept of fuzzy metric spaces in different ways.

In 1986, Jungck [3] introduced the notion of compatible maps for a pair of self-mapping. However, the study of common fixed point of non-compatible maps is also very interesting , Jungck and Rhoades[4] initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for weakly compatible maps satisfying some contractive condition in different setting such as probabilistic metric space [5, 6, 7]; fuzzy metric spaces [8, 9, 10].

In this paper, we prove a common fixed point theorem for four mapping under weakly compatible condition in fuzzy metric space. Our results substantiality generalize and improve a multitude of relevant common fixed point theorem of the existing literature in metric as well as fuzzy metric space.

II. METHODS AND MATERIAL

A. Preliminaries

Definition 2.1Let X be any set. A fuzzy set A in X is a function with domain X and values in [0, 1].

Definition 2.2[11] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if * satisfies the following condition:

(I) * is commutative and associative;
(II) * is continuous;
(III) a*1=a for all a $\in [0,1]$;
(IV) a*b ≤ c*d whenever a ≤ c and b ≤ d for all a,b,c,d $\in [0,1]$.

Kramosil and Michalek [2] introduced the concept of fuzzy metric spaces as follows:

Definition 2.3[2] The 3-tuple (X, M,*) is called a fuzzy metric space (shortly FM-space) if X is an arbitrary set * is a continuous t-norm and M is a fuzzy set in $X^2 \times$

$[0, \infty)$ satisfying the following conditions: for all x,y,z in X and s,t >0,

(I) M(x,y,0)=0,M(x,y,t)>0;
(II) M(x,y,t)=1,for all t > 0 iff x=y,
(III) M(x,y,t)=M(y,x,t),
(IV) M(x,y,t)*M(y,z,s) ≤ M(x,z,t+s)
(V) M(x,y,·):$[0,1) \rightarrow [0,1]$ is left continuous .
(VI) $\lim_{t\to\infty} M(x,y,t) = 1$ For all x, y in X.

We can fuzzify example of metric spaces into fuzzy metric space in a natural way:

Let (X,d)be a metric space .define a*b=min{a,b}for all a,b ∈ X ,we have $M(x,y,t) = \frac{t}{t+d(x,y)}$ for all x,y in X and t > 0.Then (X,M,*) is a fuzzy metric space and this fuzzy metric induced by a metric d is called the standard fuzzy metric space.

Definition 2.4[2] Let (X, M,*) be fuzzy metric space then

a) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for all t > 0 and p>0.
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ and
b) A sequence $\{x_n\}$ in X is said to be convergent to a point x ∈ X if for all
$t > 0, \lim_{n\to\infty} M(x_n, x, t) = 1$.

Definition 2.5[2] A fuzzy metric space(X,M,*) is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition2.6 [9, 12] A pair of self-mapping (A,S) of a fuzzy metric space (X,M,*) is said to be commuting if M(ASx,SAx,t)=1 for all x in X.

Definition2.7[9] A pair of self-mapping (A,S) of a fuzzy metric space (X,M,*) is said to be weakly commuting if M(ASx,SAx,t) ≥ M(Ax,Sx,t) for all x in X and t > 0.

In 1994, Mishra et.al [10] introduced the concept of compatible mapping in fuzzy metric space to concept of compatible mapping in metric space as follows:

Definition2.8 [10] A pair of self-mapping (A,S) of a fuzzy metric space (X,M,*) is said to be compatible if $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ for all t>0, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u$ for some u in X.

Definition2.9 [1] Let (X, M,*) be a fuzzy metric space A and S be self-maps on X. A point x in X is called a coincidence point of A and S iff Ax=Sx. in this case, w=Ax=Sx is called a point of coincidence of A and S.
Definition 2.10[4] A pair of self-mapping (A,S) of a fuzzy metric space (X,M,*) is said to be weakly compatible if they commute at the coincidence points. if Au=Su for some u ∈ X then ASu=SAu.

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11 Let (X, d) be a compatible metric space a∈[0,1], f:X→X a mapping such that for each x, y ∈X $\int_0^{d(fx,fy)} \varphi(t)dt \le \alpha \int_0^{d(x,y)} \varphi(t)dt$ where ψ:$R^+ \to R$ is lebesgue integral mapping which is summable $\varepsilon \geq 0$, $\int_0^{t0} \varphi(t)dt$ >0 nonnegative and such that, for each. Then f has a unique common fixed z ∈X such that for each x∈$\boldsymbol{x}$, $\lim_{n\to\infty} f^n x = z$

Lemma 2.1[2] Let $\{u_n\}$ is a sequence in a fuzzy metric space (X,M,*). if there exists a constant k ∈ (0,1)such that $M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$, n=1,2,3... Then $\{u_n\}$ is Cauchy sequence in X.

III. RESULTS AND DISCUSSION

Theorem 3.1:Let A,B,S,T be four mapping of complete fuzzy metric space (X,M,*) with condition

$\lim_{n\to\infty} M(x,y,t) = 1$ for all $x, y \in X$, $t > 0$ .k ∈ [0,1] .Let (A,T) , (B,S) be point wise weakly compatible pairs such that

1) $A(x) \subset T(x)$ and $B(x) \subset S(x)$.
2) $M(Ax, By, kt) \geq \min[M(Sx, Ax, t), M(Sx, Ty, t), M(Ty, By, t), \max\{M(Ty, Ax, t), M(Sx, By, t)\}]$
3) If one of A(x) , B(x), S(x),T(x) is complete subset of X then
   a) A and S have coincidence point .
   b) B and T have coincidence point .

then A,B,T,S have a unique common fixed point in X.
proof:

- As $A(x) \subset T(x)$ and $B(x) \subset S(x)$,
- so we can define sequence $\{x_n\}$ and $\{y_n\}$ in X. such that
  $y_{2n} = Ax_{2n} = Sx_{2n+1}$
  $y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$
- $M(Ax_{2n}, Bx_{2n+1}, kt) \geq \min[M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n}, t), \max\{M(Tx_{2n+1}, Ax_{2n}, t), M(Sx_{2n}, Bx_{2n+1}, t)\}$
- $\geq \min[M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n-1}, t), M(y_{2n-1}, y_{2n}, t), \max\{M(y_{2n}, y_{2n}, t), M(y_{2n-1}, y_{2n+1}, t)\}$
- $\geq \min[M(y_{2n-1}, y_{2n}, t), 1, M(y_{2n-1}, y_{2n}, t), \max\{1, M(y_{2n-1}, y_{2n+1}, t)\}$
- $\geq \min[M(y_{2n-1}, y_{2n}, t), 1, M(y_{2n-1}, y_{2n}, t), 1]$
- $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$
- therefore in general $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$
- $\{y_n\}$ is Cauchy sequence in X . by completeness of X. $\{y_n\}$ converges to some point z in X.
- therefore subsequence $\{y_{2n}\}$, $\{y_{2n+1}\}$, $\{y_{2n+2}\}$ converge to point Z.
- i.e $\lim_{n\to\infty} Tx_{2n+2} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = Z$
- Now suppose S(x) is complete .Let $w \in s^{-1}Z$ then Sw=Z
- $M(Aw, Bx_{2n+1}, kt) \geq \min[M(Sw, Aw, t), M(Sw, Tx_{2n+2}, t), M(Tx_{2n+2}, Bx_{2n+1}, t), \max\{M(Tx_{2n+2}, Aw, t), M(Sw, Bx_{2n+1}, t)\}]$
- $M(Aw, y_{2n+1}, kt) \geq \min[M(Z, Aw, t), M(Z, y_{2n+1}, t), M(y_{2n+1}, y_{2n+1}, t), \max\{M(Tx_{2n+1}, Aw, t), M(Z, y_{2n+1}, t)\}]$
- $n \to \infty$
- $M(Aw, Z, kt) \geq \min[M(Z, Aw, t), M(Z, Z, t), M(Z, Z, t) \max\{M(Z, Aw, t), M(Z, Z, t)\}]$
- $M(Aw, Z, kt) \geq \min[M(Z, Aw, t), 1]$
- $M(Aw, Z, kt) \geq 1$ $(\because M(Z, Aw, t) \geq 1)$
- This gives Aw=Z=Sw. therefore w is coincidence point of S and A.
- this gives $Z \in T(x)$, let $v \in T^{-1}Z \to Tv = Z$
- by equation ,
- $M(y_{2n}, Bv, kt) \geq \min[M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, Z, t), M(Z, Bv, t), \max\{M(Z, y_{2n}, t), M(y_{2n-1}, Bv, t)\}]$
- $M(y_{2n}, Bv, kt) \geq \min[M(Z, Z, t), M(Z, Z, t), M(Z, Bv, t), \max\{M(Z, Z, t), M(Z, Bv, t)\}]$
- $M(y_{2n}, Bv, kt) \geq \min[M(Z, Bv, t), 1]$
- $M(Z, Bv, kt) \geq 1$
- This gives Bv=Z=Tv so v is coincidence point of B and T .
- since (A,S) is a weakly compatible therefore A and B commute at coincidence point i.e ASw = SAw.
- this gives Az=Sz and (B,T) is weakly compatible then BTv=TBv .
- now , we will show that Az=Z=Sz by equation we have ,
- $M(Az, Bx_{2n+1}, kt) \geq \min[M(Sz, Az, t), M(Sz, Tx_{2n+2}, t), M(Tx_{2n+2}, Bx_{2n+1}, t), \max\{M(Tx_{2n+2}, Az, t), M(Sz, Bx_{2n+1}, t)\}]$
- $M(Az, y_{2n+1}, kt) \geq \min[M(Sz, Sz, t), M(Sz, y_{2n+1}, t), M(y_{2n+1}, y_{2n+1}, t), \max\{M(y_{2n+1}, Sz, t), M(Sz, y_{2n+1}, t)\}$
- $M(Az, z, kt) \geq \min[M(Sz, z, t), M(z, z, t), \max\{M(z, Sz, t), M(Sz, z, t)\}]$
- $M(Az, z, kt) \geq \min[M(Sz, z, t), 1, M(Sz, z, t)]$
- $M(Az, z, kt) \geq 1$
- this gives Az=Z=Sz ,similarly Bz=z=Tz we prove
- $M(Ax_{2n}, Bz, kt) \geq \min[M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Tz, t), M(Tz, Bz, t), \max\{M(Tz, Ax_{2n}, t), M(Sx_{2n}, Bz, t)\}$
- $M(y_{2n}, Bz, kt) \geq \min[M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, Tz, t), M(Tz, Bz, t), \max\{M(Tz, y_{2n}, t), M(y_{2n-1}, Bz, t)\}$
- $M(z, Bz, kt) \geq \min[M(z, z, t), M(z, Tz, t), M(Tz, Bz, t), \max\{M(Tz, z, t), M(z, Bz, t)\}$
- $M(z, Bz, kt) \geq \min[1, M(z, Tz, t), M(Tz, Bz, t), 1]$
- $M(z, Bz, kt) \geq M(z, Tz, t)$
- $M(z, Bz, kt) \geq 1$
- this gives Bz=z=Tz. therefore z is common fixed point of A,B,S,T.

For Uniqueness :

- Let w be another fixed point of A,B,S,T then by we have
- $M(Az, Bw, kt) \geq \min[M(Sz, Az, t), M(Sz, Tw, t), M(Tw, Bw, t), \max\{M(Tw, Az, t), M(Sz, Bw, t)\}]$
- $M(z, w, kt) \geq \min[M(z, z, t), M(z, w, t), M(w, w, t), \max\{M(w, z, t), M(z, w, t)\}]$
- $M(z, w, kt) \geq \min[1, M(z, w, t), 1, M(w, z, t)]$
- $M(z, w, kt) \geq M(w, z, t)$
- $M(z, w, kt) \geq 1$
- this gives z=w .hence z is unique common fixed point of A,B,S,T.

Theorem 3.2: Let S and T be two continuous self-mapping of a complete fuzzy metric space (X,M,*). Let A,B be two self-mapping of X satisfying $A(x) \cup B(x) \subseteq S(x) \cap T(x)$, {A,S} and {B,T} are weakly and $A(x)A(x) \subset T(x), B(x) \subset S(x)$ commuting pairs and $qM(Ax, By, kt) \geq aM(Sx, Ax, t) + bM(Ty, By, t) + cM(Sx, Ty, t) + \max\{M(Ty, Ax, t), M(Sx, By, t)\}$ for all x,y ∈ X where a,b,c < 0 and q>0 with q< a+b+c+1 and q>b ,q>c+1 .then A,B,S,T have a unique common fixed point .

proof : same as theorem 3.1.

Fixed point theorem in fuzzy metric space using integral type

Theorem: Let A,B,S,T be four mapping of complete fuzzy metric space (X,M,*) .Let (A,T) , (B,S) be point wise weakly compatible pairs .if there exist $k \in [0,1]$such that

$$\int_0^{M(Ax,By,kt)} \varphi(t)dt \geq \int_0^{\min[M(Sx,Ax,t)*M(Sx,Ty,t)*M(Ty,By,t)*\max\{M(Ty,Ax,t),M(Sx,By,t)\}]} \varphi(t)dt$$

then A,B,T,S have a unique common fixed point in X.

proof : As $A(x) \subset T(x)$ and $B(x) \subset S(x)$, so we can define sequence $\{x_n\}$ and $\{y_n\}$ in X. such that

$$y_{2n} = Sx_{2n} = Bx_{2n-1}$$
$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$$

$$\int_0^{M(Ax_{2n},Bx_{2n+1},kt)} \varphi(t)dt$$

$$\geq \int_0^{min[M(Sx_{2n},Ax_{2n},t)*M(Sx_{2n},Tx_{2n+1},t)*M(Tx_{2n+1},Ax_{2n},t)*\max\{M(Tx_{2n+1},Ax_{2n},t),M(Sx_{2n},Bx_{2n+1},t)\}]} \varphi(t)dt$$

$$\geq \int_0^{\min[M(Sx_{2n+1},Ax_{2n},t)*M(Sx_{2n},Tx_{2n+1},t)*M(Tx_{2n+1},Bx_{2n+1},t)*\max\{M(Tx_{2n+1},Ax_{2n},t),M(Sx_{2n},Bx_{2n+1},t)\}]} \varphi(t)dt$$

$$\geq \int_0^{\min[M(y_{2n+1},y_{2n+1},t)*M(y_{2n},y_{2n+1},t)*M(y_{2n+1},y_{2n+2},t)*\max\{M(y_{2n+1},y_{2n+1},t),M(y_{2n},y_{2n+2},t)\}]} \varphi(t)dt$$

$$\geq \int_0^{\min[M(y_{2n},y_{2n+1},t)*M(y_{2n+1},y_{2n+2},t)*1]} \varphi(t)dt$$

We have $\int_0^{M(y_{n+1},y_{n+2},kt)} \varphi(t)dt \geq \int_0^{M(y_n,y_{n+1},t)} \varphi(t)dt$

$\{y_n\}$ is Cauchy sequence in X . by completeness of X. $\{y_n\}$ converges to some point z in X.
since (X,M,) is complete $\{y_n\}$ converges to some point $z \in X$, and so*
$\{Ax_{2n-2}\}, \{Sx_{2n}\}$, $\{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z.
$ASx_{2n} \to Sz$ *and* $BTx_{2n-1} \to TZ$ from we get

$$\int_0^{M(ASx_{2n},BTx_{2n-1},kt)} \varphi(t)dt$$

$$\geq \int_0^{\min[M(ASx_{2n},SSx_{2n},t)*M(SSx_{2n},TTx_{2n-1},t)*M(TTx_{2n-1},BTx_{2n-1},t)*\max\{M(TTx_{2n-1},ASx_{2n},t),M(SSx_{2n},BTx_{2n-1},t)\}} \varphi(t)dt$$

$$\int_0^{M(Sz,Tz,kt)} \varphi(t)dt \geq \int_0^{min[M(Sz,Sz,t)*M(Sz,Tz,t)*M(Tz,Tz,t)*\max\{M(Tz,Sz,t),M(Sz,Tz,t)\}} \varphi(t)dt$$

$$\int_0^{M(Sz,Tz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(Sz,Tz,t)*M(Sz,Tz,t)\}} \varphi(t)dt$$

$$\int_0^{M(Sz,Tz,kt)} \varphi(t)dt \geq \int_0^{M(Sz,Tz,t)} \varphi(t)dt$$

and hence Sz=Tz

Now,

$$\int_0^{M(Az,BTx_{2n-1},kt)} \varphi(t)dt$$

$$\geq \int_0^{\min[M(Sz,Tz,t)*M(Sz,TTx_{2n-1},t)*M(TTx_{2n-1},BTx_{2n-1},t)*\max\{M(TTx_{2n-1},Az,t),M(Sz,BTx_{2n-1},t)\}]} \varphi(t)dt$$

Which implies that taking limit as $n \to \infty$

$$\int_0^{M(Az,Bz,kt)} \varphi(t)dt \geq \int_0^{min[M(Sz,Az,t)*M(Sz,Tz,t)*M(Tz,Bz,t)*\max\{M(Tz,Az,t),M(Sz,Bz,t)\}} \varphi(t)dt$$

$$\geq \int_0^{M(Az,Tz,t)} \varphi(t)dt$$

And hence Az=Tz

$$\int_0^{M(Az,Bz,kt)} \varphi(t)dt \geq \int_0^{min[M(Sz,Az,t)*M(Sz,Tz,t)*M(Tz,Bz,t)*\max\{M(Tz,Az,t),M(Sz,Bz,t)\}} \varphi(t)dt$$

$$= \int_0^{\min[M(Az,Az,t)*M(Az,Az,t)*M(Az,Bz,t)*\max\{M(Az,Az,t),M(Az,Bz,t)\}]} \varphi(t)dt$$

$$= \int_0^{M(Az,Bz,y)} \varphi(t)dt$$

so Az=Bz

it gives that Az=Bz=Tz=Sz.

Now we show that Bz=z

we get

$$\int_0^{M(Ax_{2n},Bz,kt)} \varphi(t)dt \geq \int_0^{\min[M(Sx_{2n},Ax_{2n},t)*M(Sx_{2n},Tz,t)*M(Tz,Bz,t)*\max\{M(Tz,Ax_{2n},t),M(Sx_{2n},Bz,t)\}]} \varphi(t)dt$$

$$\int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{\min[M(z,z,t)*M(z,Tz,t)*M(Tz,Bz,t)*\max\{M(Tz,z,t),M(z,Bz,t)\}]} \varphi(t)\, dt$$

$$\int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{\min[M(z,Bz,t)*M(Bz,Bz,t)*\max\{M(Bz,z,t),M(z,Bz,t)\}]} \varphi(t)$$

$$\int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{M(z,Bz,t)} \varphi(t)$$

hence Bz=z thus from z=Az=Bz=Tz=Sz and z is a common fixed point of A,B,S,T.

for uniqueness , let w be another common fixed point of A,B,S and T then

$$\int_0^{M(z,w,kt)} \varphi(t)dt = \int_0^{M(Az,Bw,kt)} \varphi(t)$$

$$\geq \int_0^{\min[M(Sz,Az,t)*M(Sz,Tw,t)*M(Tw,Bw,t)*\max\{M(Tw,Az,t),M(Sz,Bw,t)\}]} \varphi(t)dt$$

$$\geq \int_0^{M(z,w,t)} \varphi(t)dt$$

hence z=w This complete the proof of the theorem.

IV. CONCLUSION

We prove a common fixed point theorem for four mapping in weakly compatible condition in fuzzy metric space using integral type.

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