A Common Fixed Point Theorem for Weakly Compatible Mapping in Fuzzy Metric Space
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ABSTRACT
The present paper we establish a common fixed point theorem for weakly compatible pair of four maps in a fuzzy metric space.

Keywords: T-Norm, Common Fixed Point, Fuzzy Metric Space, Weakly Compatible Maps.

I. INTRODUCTION

In 1965, Zadeh \cite{1} introduced the concept of fuzzy set as a new way to represent vagueness in our everyday life. However, when the uncertain is due to fuzziness rather than randomness; it seems that the concept of a fuzzy metric space is more suitable. We can divide them into the following two groups: The first group involves those results in which a fuzzy metric on a set \( X \) is treated as a map where \( X \) represents the totality of all fuzzy points of a set and satisfy some axioms which are analogous to the ordinary metric axioms. Thus, in such an approach numerical distance are set up between fuzzy objects. On the other hand in second group, we keep those results in which the distance between objects is fuzzy and the objects themselves may or may not be fuzzy. Kramosil and Michalek \cite{2} have introduced the concept of fuzzy metric spaces in different ways.

In 1986, Jungck \cite{3} introduced the notion of compatible maps for a pair of self-mapping. However, the study of common fixed point of non-compatible maps is also very interesting, Jungck and Rhoades\cite{4} initiated the study of weakly compatible maps in metric space and showed that every pair of compatible maps is weakly compatible but reverse is not true. In the literature, many results have been proved for weakly compatible maps satisfying some contractive condition in different setting such as probabilistic metric space \cite{5, 6, 7}; fuzzy metric spaces \cite{8, 9, 10}.

In this paper, we prove a common fixed point theorem for four mapping under weakly compatible condition in fuzzy metric space. Our results substantially generalize and improve a multitude of relevant common fixed point theorem of the existing literature in metric as well as fuzzy metric space.

II. METHODS AND MATERIAL

A. Preliminaries
Definition 2.1 Let \( X \) be any set. A fuzzy set \( A \) in \( X \) is a function with domain \( X \) and values in \([0, 1]\).

Definition 2.2 \cite{11} A binary operation \( * \) is continuous t-norm if \( * \) satisfies the following condition:

(I) \( * \) is commutative and associative;
(II) \( * \) is continuous;
(III) \( a*1=a \) for all \( a \in [0,1] \);
(IV) \( a*b \leq c*d \) whenever \( a \leq c \) and \( b \leq d \) for all \( a,b,c,d \in [0,1] \).

Kramosil and Michalek \cite{2} introduced the concept of fuzzy metric spaces as follows:

Definition 2.3 \cite{2} The 3-tuple \( (X, M, *) \) is called a fuzzy metric space (shortly FM-space) if \( X \) is an arbitrary set \( * \) is a continuous t-norm and \( M \) is a fuzzy set in \( X^2 \times [0,1] \).
satisfying the following conditions: for all \(x,y,z\) in \(X\) and \(s,t > 0\),

- (I) \(M(x,y,0)=0,M(x,y,t)>0\);
- (II) \(M(x,y,t)=1\) for all \(t > 0\) iff \(x=y\),
- (III) \(M(x,y,t)=M(y,x,t)\),
- (IV) \(M(x,y,t)*M(y,z,s)=M(x,z,t+s)\),
- (V) \(M(x,y,)\) is left continuous ,
- (VI) \(\lim_{t\to\infty} M(x,y,t) = 1\) for all \(x, y\) in \(X\).

We can fuzzify example of metric spaces into fuzzy metric space in a natural way:

Let \((X,d)\) be a metric space . define \(a*b=\min\{a,b\}\) for all \(a,b \in X\), we have \(M(x,y,t) = \frac{t}{t+d(x,y)}\) for all \(x,y\) in \(X\) and \(t > 0\). Then \((X,M,*)\) is a fuzzy metric space and this fuzzy metric induced by a metric \(d\) is called the standard fuzzy metric space.

Definition 2.4[2] Let \((X,M,*)\) be fuzzy metric space then

- a) A sequence \(\{x_n\}\) in \(X\) is said to be Cauchy sequence if for all \(t > 0\) and \(p>0\),
  \[\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1\] and
- b) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) if for all \(t > 0\), \(\lim_{n\to\infty} M(x_n,x,t) = 1\).

Definition 2.5[2] A fuzzy metric space\((X,M,*)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent.

Definition 2.6 [9, 12] A pair of self-mapping \((A,S)\) of a fuzzy metric space \((X,M,*)\) is said to be commuting if \(M(ASx,SAx,t)=1\) for all \(x\) in \(X\).

Definition 2.7[9] A pair of self-mapping \((A,S)\) of a fuzzy metric space \((X,M,*)\) is said to be weakly commuting if \(M(ASx,SAx,t)\) is nonnegative and such that, for each. Then \(f\) has a unique common fixed \(z \in X\). Then \(f\) is Cauchy sequence in \(X\).

III. RESULTS AND DISCUSSION

Theorem 3.1: Let \(A,B,S,T\) be four mapping of complete fuzzy metric space \((X,M,*)\) with condition

- \(1) A(x) \subset T(x) \) and \(B(x) \subset S(x)\).
- \(2) M(Ax,By,kt) \geq \min[M(Sx,Ax,t),M(Sx,Ty,t),M(Ty,By,t),\max[M(Ty,Ax,t),M(Sx,By,t)]]\)

Definition 2.9 [1] Let \((X, M,*)\) be a fuzzy metric space \(A\) and \(S\) be self-maps on \(X\). A point \(x\) in \(X\) is called a coincidence point of \(A\) and \(S\) iff \(Ax=Sx\). in this case, \(w=Ax=Sx\) is called a point of coincidence of \(A\) and \(S\).

Definition 2.10[4] A pair of self-mapping \((A,S)\) of a fuzzy metric space \((X,M,*)\) is said to be weakly compatible if they commute at the coincidence points. if \(Au=Su\) for some \(u \in X\) then \(ASu=SUa\).

It is easy to see that two compatible maps are weakly compatible but converse is not true.

Definition 2.11 Let \((X, d)\) be a compatible metric space \(a\in[0,1], f:X\to X\) a mapping such that for each \(x, y \in X\),

\[
\int_0^t \varphi(t)dt \leq \int_0^t \psi(t)dt + k\int_0^t f(x)dt
\]

is a sequence in \(X\) such that \(\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u\) for some \(u \in X\).
As \( A(x) \subset T(x) \) and \( B(x) \subset S(x) \),
so we can define sequence \( \{x_n\} \) and \( \{y_n\} \) in \( X \). such that
\[
\begin{align*}
y_{2n} &= Ax_{2n} = Sx_{2n+1} \\
y_{2n+1} &= Bx_{2n+1} = Tx_{2n+2}
\end{align*}
\]
\( M(Ax_{2n}, Bx_{2n+1}, kt) \geq \min\{M(Sx_{2n}, Ax_{2n}, t), M(Sx_{2n}, Tx_{2n+1}, t), M(Tx_{2n+1}, Ax_{2n}, t)\} \)
\[\geq \min\{M(y_{2n-1}, y_{2n}, t), M(y_{2n-1}, y_{2n-1}, t), M(y_{2n-1}, y_{2n+1}, t)\}
\]
\max\{M(y_{2n}, y_{2n}, t), M(y_{2n-1}, y_{2n+1}, t)\}

This gives \( y_n \) is a Cauchy sequence in \( X \). by completeness of \( X \).
\{y_n\} converges to some point \( z \) in \( X \).
therefore subsequence \( \{y_{2n}\} \), \( \{y_{2n+1}\} \) converges to point \( z \).
\( n \to \infty \)
\( M(Aw, Bx_{2n+1}, kt) \geq \min\{M(Sw, Aw, t), M(Sw, Tw_{2n+2}, t), M(Tx_{2n+2}, Bx_{2n+1}, t)\}
\]
\max\{M(Tx_{2n+2}, Aw, t), M(Sw, Bx_{2n+1}, t)\}

This gives \( Aw=Z=Sw \). hence \( w \) is coincidence point of \( A \) and \( S \).
this gives \( Av=Z=Tv \) so \( v \) is coincidence point of \( B \) and \( T \).
since \( (A, S) \) is a weakly compatible therefore \( A \) and \( B \) commute at coincidence point i.e \( ASw = SAw \).
this gives \( Az=Z=Sw \) and \( (B, T) \) is weakly compatible
\( BTV=TBV \).
now , we will show that \( Az=Z=Sw \) by equation we have,
\( M(Az, Bx_{2n+1}, kt) \geq \min\{M(Sz, Az, t), M(Sz, Tx_{2n+2}, t), M(Tx_{2n+2}, Bx_{2n+1}, t),
\max\{M(Tx_{2n+2}, Az, t), M(Sw, Bx_{2n+1}, t)\}\}

This gives \( Bv=Z=Tv \) so \( v \) is coincidence point of \( B \) and \( T \).
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\max\{M(Tx_{2n+2}, Az, t), M(Sw, Bx_{2n+1}, t)\}\}

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\max\{M(Tx_{2n+2}, Az, t), M(Sw, Bx_{2n+1}, t)\}\}

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this gives \( Az=Z=Sw \) and \( (B, T) \) is weakly compatible
\( BTV=TBV \).
now , we will show that \( Az=Z=Sw \) by equation we have,
\( M(Az, Bx_{2n+1}, kt) \geq \min\{M(Sz, Az, t), M(Sz, Tx_{2n+2}, t), M(Tx_{2n+2}, Bx_{2n+1}, t),
\max\{M(Tx_{2n+2}, Az, t), M(Sw, Bx_{2n+1}, t)\}\}

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this gives \( Az=Z=Sw \) and \( (B, T) \) is weakly compatible
\( BTV=TBV \).
now , we will show that \( Az=Z=Sw \) by equation we have,
Theorem 3.2: Let S and T be two continuous self-mapping of a complete fuzzy metric space \((X,M,\ast)\). Let A,B be two self-mapping of \(X\) satisfying \(A(x) \cup B(x) \subseteq S(x)\cap T(x)\), \(\{A,S\}\) and \(\{B,T\}\) are weakly and \(A(x)A(x) \subseteq T(x),B(x) \subseteq S(x)\) commuting pairs and \(qM(Ax,By,kt) \ge aM(Sx, Ax, t) + bM(Ty, By, t) + cM(Sx, Ty, t) + \max\{M(Ty, Ax, t), M(Sx, By, t)\}\) for all \(x,y \in X\) where \(a,b,c < 0\) and \(q > 0\) with \(q > a+b+c+1\) and \(q > b, q > c+1\) then A,B,S,T have a unique common fixed point.

**proof:** same as theorem 3.1.

Fixed point theorem in fuzzy metric space using integral type

Theorem: Let A,B,S,T be four mapping of complete fuzzy metric space \((X,M,\ast)\). Let \((A,T)\), \((B,S)\) be point wise weakly compatible pairs. If there exist \(k \in [0,1]\) such that

\[
\int_0^M(Ax,By,kt) \varphi(t)dt \ge \int_0^M[A(x,ty,y)\cdot M(x,Ty,t)+M(Ty,By,t)+\max\{M(Ty,Ax,t),M(Sx,By,t)\}] \varphi(t)dt
\]

then A,B,T,S have a unique common fixed point in \(X\).

**proof:** As \(A(x) \subseteq T(x)\) and \(B(x) \subseteq S(x)\), so we can define sequence \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that

\[
y_{2n} = Sx_{2n} = Bx_{2n-1}
y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}
\]

\[
M(Ax_{2n},Bx_{2n+1},kt) = \varphi(t)dt
\]

\[
\int_0^M\min\{M(Sx_{2n},Ax_{2n},t)+M(Sx_{2n},Tx_{2n+1},t)+M(Tx_{2n+1},Ax_{2n},t)+\max\{M(Tx_{2n+1},Ax_{2n},t),M(Sx_{2n},Bx_{2n+1},t)\}\} \varphi(t)dt
\]

\[
\int_0^M\min\{M(Sx_{2n+1},Ax_{2n},t)+M(Sx_{2n},Tx_{2n+1},t)+M(Tx_{2n+1},Bx_{2n+1},t)+\max\{M(Tx_{2n+1},Ax_{2n},t),M(Sx_{2n},Bx_{2n+1},t)\}\} \varphi(t)dt
\]

\[
\int_0^M\min\{M(y_{2n+1},y_{2n+1},t)+M(y_{2n},y_{2n+1},t)+M(y_{2n+1},y_{2n+2},t)+\max\{M(y_{2n+1},y_{2n+1},t),M(y_{2n},y_{2n+2},t)\}\} \varphi(t)dt
\]

\[
\int_0^M\min\{M(y_{2n+1},y_{2n},t)+M(y_{2n+1},y_{2n+2},t)+M(y_{2n+1},y_{2n+2},t)+1\} \varphi(t)dt
\]

We have \(\int_0^M(y_{n+1},y_{n+2},kt) \varphi(t)dt \ge \int_0^M(y_n,y_{n+1},t) \varphi(t)dt\)

\(\{y_n\}\) is Cauchy sequence in \(X\). by completeness of \(X\). \(\{y_n\}\) converges to some point \(z\) in \(X\).

since \((X,M,\ast)\) is complete \(\{y_n\}\) converges to some point \(z \in X\), and so \(\{Ax_{2n-2}\},\{Sx_{2n}\},\{Bx_{2n-1}\}\) and \(\{Tx_{2n-1}\}\) also converges to \(z\).

\(ASx_{2n} \rightarrow Sz\) and \(BTx_{2n-1} \rightarrow Tz\) from we get

\[
\int_0^M[\min\{M(ASx_{2n},SSx_{2n},t)+M(SSx_{2n},TTx_{2n-1},t)+M(TTx_{2n-1},ASx_{2n},t)+\max\{M(TTx_{2n-1},ASx_{2n},t),M(SSx_{2n},BTx_{2n-1},t)\}\}] \varphi(t)dt
\]

\[
\int_0^M[M(Sz,Sz,kt)\le \min\{M(Sz,Sz,t)+M(Tz,Sz,t)+M(Tz,Tz,t)+\max\{M(Tz,Sz,t),M(Sz,Tz,t)\}\}] \varphi(t)dt
\]

\[
\int_0^M[M(Sz,Sz,kt)\le \min\{M(Sz,Tz,t)+M(Sz,Sz,t)\}] \varphi(t)dt
\]
and hence $S_z = T_z$

Now,
\[ \int_0^{M(Az,BT_xe_{2n-1},kt)} \varphi(t)dt \]
\[ \geq \int_0^{\min\{M(Sz,Tz,t)\ast M(Sz,TTx_{2n-1},t)\ast M(TTx_{2n-1},BTx_{2n-1},t)\ast \max\{M(TTx_{2n-1},Az,t)\ast M(Sz,BTx_{2n-1},t)\}} \varphi(t)dt \]

Which implies that taking limit as $n \to \infty$
\[ \int_0^{M(Az,Bz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(Sz,Az,t)\ast M(Sz,Tz,t)\ast M(Tz,Bz,t)\ast \max\{M(Tz,Az,t)\ast M(Sz,Bz,t)\}} \varphi(t)dt \]
\[ \geq \int_0^{M(Az,Tz,t)} \varphi(t)dt \]

And hence $A_z = T_z$
\[ \int_0^{M(Az,Bz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(Az,Az,t)\ast M(Az,Tz,t)\ast M(Az,Bz,t)\ast \max\{M(Az,Az,t)\ast M(Az,Bz,t)\}} \varphi(t)dt \]
\[ = \int_0^{M(Az,Bz,y)} \varphi(t)dt \]

so $A_z = B_z$

it gives that $A_z = B_z = T_z = S_z$.

Now we show that $B_z = z$

we get
\[ \int_0^{M(Ax_{2n},Bz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(Sx_{2n},Ax_{2n},t)\ast M(Sx_{2n},Tz,t)\ast M(Tz,Bz,t)\ast \max\{M(Tz,Ax_{2n},t)\ast M(Sx_{2n},Bz,t)\}} \varphi(t)dt \]
\[ \int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(z,z,t)\ast M(z,Tz,t)\ast M(Tz,Bz,t)\ast \max\{M(Tz,z,t)\ast M(z,Bz,t)\}} \varphi(t)dt \]

\[ \int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{\min\{M(z,Bz,t)\ast M(Bz,Bz,t)\ast \max\{M(Bz,z,t)\ast M(z,Bz,t)\}} \varphi(t) \]
\[ \int_0^{M(z,Bz,kt)} \varphi(t)dt \geq \int_0^{M(z,Bz,t)} \varphi(t) \]

hence $B_z = z$ thus from $z = A_z = B_z = T_z = S_z$ and $z$ is a common fixed point of $A, B, S, T$.

for uniqueness, let $w$ be another common fixed point of $A, B, S$ and $T$ then
\[ \int_0^{M(z,w,kt)} \varphi(t)dt = \int_0^{M(Az,Bw,kt)} \varphi(t) \]
\[ \geq \int_0^{\min\{M(Sz,Az,t)\ast M(Sz,Tw,t)\ast M(Tw,Bw,t)\ast \max\{M(Tw,Az,t)\ast M(Sz,Bw,t)\}} \varphi(t)dt \]
\[ \geq \int_0^{M(z,w,t)} \varphi(t)dt \]

hence $z = w$ This complete the proof of the theorem.
IV. CONCLUSION

We prove a common fixed point theorem for four mapping in weakly compatible condition in fuzzy metric space using integral type.

V. REFERENCES