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Orthogonal Generalized Derivation and Left (Resp. Right) Centralizers on Semiprime Ring

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ABSTRACT

The object of this article is present the orthogonally of different two concepts. We proved: Assume $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog gd and l(rr)c on 2-tortion free semiprime ring $\dot{R}$ s.t. $\dot{t}$ is commutative then $\dot{d}$ and $\dot{t}$ are orthog where $\dot{d}$ is associated derivation of $\dot{\mathfrak{D}}$.

Keywords: Derivation, Generalized Derivation, Left Centralizer, Orthogonal, Semiprime Ring.

I. INTRODUCTION

The orthogonal is one of important concept in algebra[2,3] where in this paper we study the orthogonal of left (resp. right)-centralizer and derivation on ring . we prove that

Let R be semiprime ring, d and t

are derivation and left (resp.righr)-centralizer on R respectively then d and t are orthogonal iff ∃ a,b∈R, b∈Z(R);s.t. dt(u) = au+ ub, for all u∈$\dot{R}$.

The following result we needed later

Lemma 1.1:[1] Assume $\dot{R}$ with char. ≠2, semi-prime and $\dot{a},\dot{b} \in \acute{\mathcal{R}}$ what follows is equivalent

$\forall \dot{c} \in \dot{\mathcal{R}}$,

i. $\dot{a}\dot{c}\dot{b} = 0$

ii. $\dot{b}\dot{c}\dot{a} = 0$

iii. $\dot{a}\dot{c}\dot{b} + \dot{b}\dot{c}\dot{a} = 0$

If one of above conditions is satisfy then $\acute{b}\,\acute{a}$=0 =$\acute{a}\,\acute{b}$.

II. ORTHOGONAL GENERALIZED DERIVATION AND LEFT (RESP. RIGHT)-CENTRALIZER OF RING

Definition 2.1: Assume $\dot{D}$ is a generalized derivation (for shout gd) on a ring $\dot{\mathcal{R}}$ and $\dot{t}$ be left (resp.right)-centralizer (for shout l(rr)c)on $\dot{\mathcal{R}}$ then $\dot{d}$ and $\dot{t}$ is orthogonal (for shout orthog) iff

$\forall \dot{a},\dot{b} \in \acute{\mathcal{R}}$, $\dot{D}(\dot{a})\dot{\mathcal{R}}\dot{t}(\dot{b}) = 0 = \dot{t}(\dot{a})\dot{R}\dot{D}(\dot{b})$

Example2.2: Assume $\acute{\mathcal{R}}$ be a ring $\dot{S} = \acute{\mathcal{R}} \times \acute{\mathcal{R}}$, define derivation $\dot{D}_1$ and l (r r)c $\dot{t}$ on$\acute{S}$.

If $\dot{D}$ be defined on $\dot{S}$×$\dot{S}$ by $\dot{D}((\dot{a},\dot{b}),(\dot{a}_1,\dot{b}_1))$= $(\dot{D}_1(\dot{a},\dot{b}),0)$, then $\dot{D}$ is derivation on $\dot{S}$x$\dot{S}$ associated with derivation $\dot{d}$ on Ṡ × $\dot{S}$ defined by $\dot{d}((\dot{a},\dot{b}),(\dot{a}_1,\dot{b}_1))$= $(\dot{d}_1(\dot{a},\dot{b}), 0)$ s.t. $\dot{d}_1$ is derivation on $\dot{S}$.

and $\dot{T}$ is defined on $\dot{S}$×$\dot{S}$ by

T(($\acute{a}$, $\dot{b}$),($\dot{u}_1$, $\dot{v}_1$))= (0 , $\dot{t}$ ($\dot{a}_1$, $\dot{b}_1$)), then $\dot{T}$ is left (resp.right)-centralizer on $\dot{S} \times \dot{S}$. Hence $\dot{D}$ and $\dot{T}$ are orthog on $\dot{S}$×$\dot{S}$.

Example2.3: Assume $\dot{\mathcal{R}} = \left\{\begin{bmatrix} o & a \\ o & o \end{bmatrix} : a \in \mathcal{Z}\right\}$ be a ring defines $\dot{\boldsymbol{D}}, \dot{\boldsymbol{t}}$ on $\dot{\mathcal{R}}$ by

$\dot{D}\begin{bmatrix} o & a \\ o & o \end{bmatrix} = \begin{bmatrix} o & 8a \\ o & o \end{bmatrix}$ associated with derivation $\dot{d}$ on $\dot{\mathcal{R}}$ by

$$\dot{d}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3a^4 \\ 0 & 0 \end{bmatrix}$$

$$\dot{t}\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 5a^7 \\ 0 & 0 \end{bmatrix}; \forall \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \in \dot{\mathcal{R}}$$

Then $\dot{\mathfrak{D}}$ and $\acute{t}$ are orthog.

Lemma 1: Let $\dot{\mathfrak{D}}$ and $\dot{t}$ be gd and l(rr)c on a semiprime ring $\dot{R}$ is orthog iff

$\dot{\mathfrak{D}}(\acute{a})\,\acute{t}(\grave{b}) + \dot{t}(\grave{b})\dot{\mathfrak{D}}(\dot{a})=0$, $\forall \acute{a},\dot{b}\in\acute{\mathcal{R}}$.

Proof: Assume $\acute{\mathfrak{D}}$ and $\dot{t}$ are orthog

To prove $\dot{\mathfrak{D}}(\dot{a})\,\dot{t}(\dot{b}) + \dot{t}(\acute{b})\,\dot{\mathfrak{D}}(\acute{a})=0$, for all $\acute{a},\dot{b}\in\dot{R}$.

Since $\acute{d}$ and $\acute{t}$ are orthog

$\dot{\mathfrak{D}}(\acute{a})\dot{w}\dot{t}(\dot{b})=0= \dot{t}(\dot{b})\dot{w}\dot{\mathfrak{D}}(\dot{a})$

By Lemma 1.1 we get

$\dot{\mathfrak{D}}(\acute{a})\,\dot{t}(\acute{b}) = \dot{t}(\acute{b})\,\dot{\mathfrak{D}}(\dot{a})$

$\acute{\mathfrak{D}}(\dot{\mathfrak{D}})\dot{t}(\acute{b}) + \dot{t}(\acute{b})\dot{\mathfrak{D}}(\dot{a})=0$

Conversely $\acute{\mathfrak{D}}(\acute{a})\dot{t}(\dot{b}) + \dot{t}(\dot{b})\dot{\mathfrak{D}}(\dot{a})=0$

Replace $\acute{b}$ by $\acute{b}\acute{a}$

$\dot{\mathfrak{D}}(\dot{a})\dot{t}(\dot{b}\dot{a})+\dot{t}(\dot{b}\dot{a})\dot{\mathfrak{D}}(\dot{a}) = 0$

$\dot{\mathfrak{D}}(\dot{a})\,\dot{t}(\dot{b})\acute{a} + \dot{t}(\acute{b})\,\acute{a}\dot{\mathfrak{D}}(\acute{a})=0$

$\dot{t}(\acute{b})\dot{a}\dot{\mathfrak{D}}(\dot{a})=0$

by Lemma1.1, $\dot{\mathfrak{D}}(\dot{a})\dot{a}\dot{t}(\dot{b}) + \dot{t}(\dot{b})\dot{a}\dot{\mathfrak{D}}(\dot{a})=0$

hence $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog

III.MAIN RESULTS

Lemma 2: Let $\dot{\mathfrak{D}}$ and $\dot{t}$ be gd and l(rr)c on a semiprime ring $\dot{R}$ then $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog. if and only if $\dot{\mathfrak{D}}\dot{t}=0$.

Proof: suppose that $\dot{\mathfrak{D}}\dot{t}=0$ and $\dot{a},\dot{b}\in\dot{R}$

$0= \dot{\mathfrak{D}}\dot{t}(\dot{a}\dot{b})$

$= \acute{\mathfrak{D}}(\acute{t}(\acute{a})\acute{b})$

$= \dot{\mathfrak{D}}(\acute{t}(\dot{a}))\acute{b} + \dot{t}(\dot{a})\acute{d}(\dot{b})$

$= \dot{t}(\acute{a})\dot{d}(\acute{b})$

Replace $\dot{a}$ by $\acute{a}\dot{w}$

$0= \dot{t}(\dot{a})\dot{w}\dot{d}(\dot{b})$

By lemma1.1; $0= \acute{t}(\acute{a})\dot{w}\dot{d}(\acute{b}) + \acute{d}(\dot{b})\dot{w}\dot{t}(\acute{a})$

$\Longrightarrow$ $\dot{d}$ and $\acute{t}$ are orthog

Conversely let $\acute{\mathfrak{D}}$ and $\acute{t}$ are orthog

$\dot{\mathfrak{D}}(\ddot{a})\dot{\omega}\dot{t}(\acute{b})= 0$

$\acute{\mathfrak{D}}(\dot{\mathfrak{D}}(\dot{a})\acute{w}\dot{t}(\acute{b}))=0$

$\acute{\mathfrak{D}}(\dot{\mathfrak{D}}(\acute{a}))\,\dot{d}(\dot{w})\dot{d}(\dot{t}(\dot{b}))=0$

Replace $\dot{\mathfrak{D}}(\dot{\mathfrak{D}}(\dot{a}))$ by $\dot{\mathfrak{D}}(\dot{t}(\acute{b}))$ *and* $\dot{t}(\acute{t}(\dot{b}))$ by $\dot{\mathfrak{D}}(\dot{t}(\acute{b}))$

$\dot{\mathfrak{D}}(\acute{t}(\dot{b}))\dot{R}\dot{\mathfrak{D}}(\dot{t}(\dot{b})) = 0$

By semiprimeness, $\acute{\mathfrak{D}}(\dot{t}(\dot{b}))=0$; for all $\dot{b}\in\dot{R}$, $\acute{d}\acute{t}=0$.

Lemma 3: Let $\dot{\mathfrak{D}}$ and $\dot{t}$ be gd and l(rr)c on a semiprime ring $\dot{R}$, of char$\neq$2, then $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog iff $\dot{\mathfrak{D}}\acute{t} + \dot{t}\dot{\mathfrak{D}} = 0$.

Proof: assume that $\dot{\mathfrak{D}}\acute{t}+\dot{t}\dot{\mathfrak{D}}=0$; $\forall$ $\dot{a},\dot{b}\in\dot{R}$

$0= (\dot{\mathfrak{D}}\acute{t}+\dot{t}\dot{\mathfrak{D}})(\dot{a}\dot{b})$

$= \acute{\mathfrak{D}}(\dot{t}(\dot{a}\dot{b})) + \dot{t}(\dot{\mathfrak{D}}(\dot{a}\dot{b}))$

$= \dot{\mathfrak{D}}(\acute{t}(\dot{a})\dot{b}) + \dot{t}(\acute{\mathfrak{D}}(\dot{a})\dot{b} + \dot{a}\dot{\mathfrak{D}}(\acute{b}))$

$= \acute{\mathfrak{D}}(\dot{t}(\dot{a}))\acute{b} + \dot{t}(\dot{a})\dot{\mathfrak{D}}(\dot{b}) + \dot{t}(\dot{\mathfrak{D}}(\dot{a}))\acute{b} + \dot{t}(\dot{a})\,\dot{\mathfrak{D}}(\dot{b})$

$= 2\,\dot{t}(\acute{a})\,\acute{\mathfrak{D}}(\dot{b})$

By 2-torsion free, $\dot{t}(\dot{a})\dot{\mathfrak{D}}(\dot{b})=0$

by result1.1; $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog.

Conversely; let $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog, by Lemma 2 we get $\dot{\mathfrak{D}}\acute{t}+\dot{t}\dot{\mathfrak{D}}=0$.

Lemma 4: Assume $\dot{R}$ is semiprime of char.$\neq$2 if $\dot{\mathfrak{D}}^2=\dot{t}^2$, *then* $\dot{\mathfrak{D}}$- $\dot{t}$ and $\dot{\mathfrak{D}}+\acute{t}$ are orthog.

Proof: $((\dot{\mathfrak{D}} - \acute{t})(\acute{\mathfrak{D}} + \dot{t}) + (\dot{\mathfrak{D}}+\acute{t})(\dot{\mathfrak{D}}-\acute{t}))(\dot{z})$

$= (\acute{\mathfrak{D}} - \dot{t})(\dot{z})(\dot{\mathfrak{D}}+\acute{t})(\dot{z}) + (\acute{\mathfrak{D}} + \dot{t})(\dot{z})(\acute{\mathfrak{D}} - \dot{t})(\dot{z})$

$=(\tilde{\mathfrak{D}}(\dot{z}) - \acute{t}(\dot{z}))(\acute{\mathfrak{D}}(\dot{z}) + \dot{t}(\dot{z})) + (\acute{\mathfrak{D}}(\dot{z}) + \acute{t}(\acute{z}))(\dot{\mathfrak{D}}(\acute{z}) - \acute{t}(\dot{z}))$

$= \dot{\mathfrak{D}}^2(\acute{z}) + \dot{\mathfrak{D}}(\dot{z})\,\dot{t}(\acute{z}) - \dot{t}(\acute{z})\,\dot{\mathfrak{D}}(\dot{z}) - \acute{\mathfrak{D}}^2(\acute{z}) + \acute{\mathfrak{D}}^2(\ddot{z}) - \acute{\mathfrak{D}}(\acute{z})\dot{t}(\dot{z}) + \dot{t}(\dot{z})\,\acute{\mathfrak{D}}(\dot{z}) - \dot{t}^2(\acute{z})$

$=0$

By Lemma1; $\acute{\mathfrak{D}}+\dot{t}$ and $\dot{\mathfrak{D}}-\acute{t}$ are orthog.

Lemma 5: Assume $\dot{R}$ be semi-prime ring ; $\dot{\mathfrak{D}}$, $\dot{t}$ are gd and l(rr)c on $\dot{R}$ s.t. $\acute{d}^2=\dot{t}^2$, then $\dot{d}$-$\dot{t}$ and $\dot{d}+\acute{t}$ are orthog. Where $\dot{d}$ is associated derivation of $\dot{D}$.

Proof: $((\acute{d}-\dot{t})(\dot{d}+\acute{t}) + (\dot{d}+\dot{t})(\grave{d}-\acute{t}))(\acute{a})$

$= (\grave{d}-\acute{t})(\acute{a})(\grave{d}+\acute{t})(\acute{a}) + (\dot{d}+\acute{t})(\dot{a})(\grave{d}-\acute{t})\acute{a})$

$= (\grave{d}(\dot{a}) - \acute{t}(\dot{a}))(\dot{d}(\acute{a}) + \acute{t}(\dot{a})) + (\grave{d}(\acute{a}) + \dot{t}(\acute{a}))(\grave{d}(\dot{a}) - \dot{t}(\acute{a}))$

$= \acute{d}^2(\dot{a}) + \dot{d}(\dot{a})\dot{t}(\dot{a}) - \dot{t}(\acute{a})\dot{d}(\acute{a}) - \dot{d}^2(\dot{a}) + \acute{d}^2(\dot{a}) - \grave{d}(\dot{a})\,\acute{t}(\dot{a}) + \acute{t}(\dot{a})\dot{d}(\acute{a}) - \dot{t}^2(\acute{a})$

$= 0$

By Lemma1; $\acute{d}+\acute{t}$ and $\dot{d}$-$\dot{t}$ are orthog.

Lemma 6: Assume $\dot{R}$ be semiprime ring then $\dot{d}$ and $\dot{t}$ are orthog iff $\exists\ c, \acute{e} \in \dot{\mathcal{R}}$,
$\acute{e} \in Z(\dot{\mathcal{R}})$; s. t. $\dot{\mathfrak{D}}\dot{t}(\dot{a}) = c\dot{a} + \dot{a}\dot{e}$, $\forall\ \acute{a} \in \dot{\mathcal{R}}$.
Proof : Let $\dot{\mathfrak{D}}\dot{t}(\dot{a}) = c\dot{a} + \dot{a}\dot{e}$
Replace $\dot{a}$ by $\dot{a}\dot{b}$
$\dot{\mathfrak{D}}\dot{t}(\dot{a}\dot{b}) = c\dot{a}\dot{b} + \dot{a}\dot{b}\dot{e}$
$\dot{\mathfrak{D}}(\dot{t}(\acute{a})\dot{b}) = c\dot{a}\dot{b} + \dot{a}\dot{b}\dot{e}$
$\dot{\mathfrak{D}}\dot{t}(\acute{a})\acute{b} + \acute{t}(\dot{a})\dot{d}(\dot{b}) = \dot{c}\dot{a}\dot{b} + \dot{a}\dot{b}\dot{e}$
$\dot{c}\dot{a}\dot{b} + \dot{a}\dot{e}\dot{b} + \acute{t}(\dot{a})\dot{d}(\dot{b}) = c\dot{a}\dot{b} + \dot{a}\dot{b}\dot{e}$
$\dot{a}\dot{e}\dot{b} + \acute{t}(\dot{a})\dot{d}(\dot{b}) = \dot{a}\dot{b}\dot{e}$
replace $\dot{b}$ by $\dot{b}\dot{a}$
$\acute{a}\dot{e}\dot{b}\dot{a} + \dot{t}(\dot{a})\dot{d}(\dot{b}\dot{a}) = \dot{a}\dot{b}\dot{a}\dot{e}$
$\acute{a}\dot{e}\dot{b}\dot{a} + \dot{t}(\dot{a})\,\dot{d}(\dot{b})\dot{a} + \dot{t}(\dot{a})\dot{b}d(\dot{a}) = \dot{a}\dot{b}\dot{a}\dot{e}$
$\dot{a}\dot{b}\dot{e}\dot{a} + \dot{t}(\dot{a})\dot{b}\acute{d}(\dot{a}) = \dot{a}\dot{b}\dot{e}\dot{a}$
$\grave{t}(\acute{a})\dot{b}\dot{d}(\dot{a}) = 0$
by lemma 1.1 ; $\acute{\mathfrak{D}}$and $\dot{t}$ are orthog.
Conversely, since $\acute{\mathfrak{D}}$ and $\dot{t}$ are orthogo, $\dot{\mathfrak{D}}\dot{t} = 0$, so we can choose $c = \dot{e} = 0$ so that
$\dot{\mathfrak{D}}\dot{t}(\dot{x}) = c\dot{x} + \dot{x}\acute{e}$.

Lemma 7: Assume $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthog gd and l(rr)c on 2-tortion free semiprime ring $\dot{R}$ s.t. $\dot{t}$ is commutative then $\dot{d}$ and $\dot{t}$ are orthog where $\dot{d}$ is associated derivation of $\dot{\mathfrak{D}}$.
Proof: Assume $\dot{\mathfrak{D}}$ and $\dot{t}$ are orthogonal
$\Longrightarrow \dot{\mathfrak{D}}(\dot{a})\dot{R}\dot{t}(\dot{b}) = (0)$
By lemma 1.1 $\Longrightarrow \grave{\mathfrak{D}}(\dot{a})\ \dot{t}(\dot{b}) = 0$
Since $\grave{t}$ is commutative $\Longrightarrow$ $\grave{t}$ ($\dot{b}$) $\grave{\mathfrak{D}}(\dot{a}) = 0$. ...(1)
Replace $\dot{w}\dot{a}$ by $\dot{a}$
$\Longrightarrow \dot{t}(\dot{b})\ \dot{\mathfrak{D}}(\dot{w}\dot{a}) = 0$
$\Longrightarrow \dot{t}(\acute{b})\ \dot{\mathfrak{D}}(\dot{w})\dot{a} + \dot{t}(\dot{b})\dot{w}\dot{d}(\dot{a}) = 0$
$\Longrightarrow \dot{t}(\dot{b})\dot{w}\dot{d}(\dot{a}) = 0$ by (1)
$\Longrightarrow \dot{d}(\dot{a})\dot{w}\dot{t}(\dot{b}) = 0$. By commutative of $\dot{t}$
$\Longrightarrow \dot{d}$ and $\dot{t}$ are orthog.

IV.CONCLUSION

The concept of orthogonal is the important topic in algebra. The concept of orthogonality between two different types of mappings is to generalize the definitions of orthognol of same typs of mappings on ring and to Specify the study between two concepts: Orthogonal generalized derivation and left (resp. right) centralizer as well as orthogonal between derivation and left (resp. right) centralizer.

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