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Subclass of Bi-Univalent Functions Using Salagean Derivative Operator

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ABSTRACT

Here we introduce a new subclass of the function class $\sum$ of bi-univalent functions defined by convolution using Salegean operator in the open unit disc. Furthermore, we obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions of this class. Relevant connections of the results presented here with various well-known results are briefly indicated.

Keywords: Analytic; Univalent; Bi-univalent; Salagean derivative; convolution; starlike and convex functions; coefficients bounds.

I. INTRODUCTION

Let A denote the class of the functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

Which are analytic in the open unit disk and $U = \{ z \in C : |z| < 1 \}$ satisfy the normalization condition $f(0) = f^{'}(0) - 1 = 0$. Let S be the subclass of A consisting of functions of the form (1.1) which are also univalent in U.

For $f(z)$ defined by (1.1) and $\Phi(z)$ defined by $\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \quad (\phi_n \geq 0)$ (1.2)

The Hadamard product $(f * \Phi)(z)$ of the function $f(z)$ and $\Phi(z)$ defined by

$$(f * \Phi)(z) = z + \sum_{n=2}^{\infty} a_n \phi_n z^n \qquad (1.3)$$

For $n \in N_0, 0 \leq \beta < 1, \lambda \geq 0$, we introduce the subclass $Q(n, \lambda, \beta)$ of S of functions of the form (1.1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n \quad (h_n > 0) \qquad (1.4)$$

and satisfying the condition

$$\operatorname{Re}\left\{\frac{(1-\lambda)D^n(f*h)(z)+\lambda D^{n+1}(f*h)(z)}{z}\right\}>\beta,\ z\in U \quad (1.5)$$

Where D^n stands for Salagean derivative introduced by Salegean [1].

For $n=0$ it reduces to the class $Q_\lambda(\beta)$ studied by Ding et al. [2], (see also [3-6]).

It is well known that every $f\in S$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z,\quad (z\in U)\quad \text{and}\quad f^{-1}(f(w))=w,\quad \left(|w|<r_0(f), r_0(f)\geq\frac{1}{4}\right),$$

Where,

$$f^{-1}(w)=w-a_2w^2+\left(2a_2^2-a_3\right)w^3 -\left(5a_2^3-2a_2a_3+a_4\right)w^4+\ldots$$

A function $f(z)\in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U.

Let $\sum$ denote the class of bi-univalent functions in U given by (1.1). For more basic results one may refer Srivastava et al.[7] and references there in.

Brannan and Taha [8] (see also [9]) introduced certain subclasses of the bi-univalent function class $\sum$ similar to the familiar subclasses $S^*(\alpha)$ and $K(\alpha)$ of starlike and convex functions of order $\alpha(0\leq\alpha<1)$ respectively (see [10]). Thus, following Brannan and Taha [8] (see also [9]), a function $f(z)\in A$ is in the class $S_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order $\alpha(0\leq\alpha<1)$ if each of the following conditions is satisfied

$$f\in\Sigma\ \textit{and}\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2},\ (0\leq\alpha<1, z\in U)\ \text{and}$$

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right|<\frac{\alpha\pi}{2},\ (0\leq\alpha<1, w\in U)$$

Where g is the extension of f^{-1} on U. Similarly, a function $f\in A$ is in the class $K_\Sigma(\alpha)$ of strongly bi-convex functions of order $\alpha(0\leq\alpha<1)$ if each of the following conditions are satisfied

$$f\in\Sigma\ \textit{and}\ \left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2},\ (0\leq\alpha<1, z\in U)\ \text{and}$$

$$\left|\arg\left(1+\frac{wg''(w)}{g'(w)}\right)\right|<\frac{\alpha\pi}{2},\ (0\leq\alpha<1, w\in U)$$

Where g is the extension of f^{-1} on U. The classes $S_\Sigma^*(\alpha)$ and $K_\Sigma(\alpha)$ of bi-starlike and bi-convex functions of order α, corresponding to the function classes $S^*(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S_{\Sigma}^{*}(\alpha)$and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$and $|a_3|$ (for detail,see [8,9]).

Recently, several researchers such as ([7,11-13]) obtained the coefficients$|a_2|$, $|a_3|$ of bi-univalent functions for the various subclasses of the function class Σ. Motivating with their work, we introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$and $|a_3|$ for functions in these new subclass of the function class Σ employing the techniques used earlier by Srivastava et al. [7] and Frasin and Aouf[11].

In order to prove our main results, we require the following lemma due to [14].

Lemma1.1. If $h \in P$ then $|c_k| \le 2$ for each k, where P is the family of all functions h analytic in U for which

$\operatorname{Re}\{h(z)\} > 0,$

$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$ for $z \in U$

II. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $B_{\Sigma}(n,\alpha,\lambda)$

Definition2.1. A function $f(z)$ given by (1.1) is said to be in the class $B_{\Sigma}(n,\alpha,\lambda)$ if the following conditions are satisfied:

$f \in \Sigma$ *and*

$$\left|\arg\left\{\frac{(1-\lambda)D^n(f*h)(z)+\lambda D^{n+1}(f*h)(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}$$

$$;(0 \le \alpha < 1, \lambda \ge 1,\ z \in U) \quad (2.1)$$

$f \in \Sigma$ *and*

$$\left|\arg\left\{\frac{(1-\lambda)D^n(f*h)^{-1}(z)+\lambda D^{n+1}(f*h)^{-1}(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}$$

$$;(0 \le \alpha < 1, \lambda \ge 1,\ z \in U) \quad (2.2)$$

where the function $h(z)$ is given by (1.4) and $(f*h)^{-1}(w)$ is defined by:

$$(f*h)^{-1}(w) = w - a_2 h_2 w^2 + \left(2a_2^2 h_2^2 - a_3 h_3\right)w^3$$
$$-\left(5a_2^3 h_2^3 - 5a_2 h_2 a_3 h_3 + a_4 h_4\right)w^4 + \ldots \quad (2.3)$$

We note that for $n = 0, \lambda = 1$ the class $B_{\Sigma}(n,\alpha,\lambda)$ reduces to the class H_{Σ}^{α} introduced and studied by Srivastava et al. [7] and for $n = 0$ the class $B_{\Sigma}(n,\alpha,\lambda)$reduces to the class $B_{\Sigma}(\alpha,\lambda)$ introduced and studied by Frasin and Aouf [11]. We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $B_{\Sigma}(n,\alpha,\lambda)$.

Theorem 2.1. Let the function $h(z)$ is given by (1.1) be in the class $B_{\Sigma}(n,\alpha,\lambda)$, $n \in N_0, 0 < \alpha \le 1$ and $\lambda \ge 1$.Then

$$|a_2| \le \frac{2\alpha}{h_2\sqrt{4^n(1+\lambda)^2 + \alpha\left(2.3^n(1+2\lambda) - 4^n(1+\lambda)^2\right)}} \quad (2.4)$$

and

$$|a_3| \le \frac{1}{h_3}\left(\frac{2\alpha}{(1-\lambda)3^n + \lambda 3^{n+1}} + \frac{4\alpha^2}{\left[(1-\lambda)2^n + \lambda 2^{n+1}\right]^2}\right) \quad (2.5)$$

Proof. It follows from (2.1) and (2.2) that

$$\frac{\left[(1-\lambda)D^n\left(f*h\right)(z)+\lambda D^{n+1}\left(f*h\right)(z)\right]}{z}=\left[p(z)\right]^{\alpha}, \quad (2.6)$$

and

$$\frac{\left[(1-\lambda)D^n\left(f*h\right)^{-1}(w)+\lambda D^{n+1}\left(f*h\right)^{-1}(w)\right]}{w}=\left[q(w)\right]^{\alpha} \quad (2.7)$$

Where $p(z)$ and $q(z)$ in P and have the forms

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\ldots \qquad (2.8)$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\ldots \qquad (2.9)$$

Now, equating the coefficient in (2.6) and (2.7), we obtain

$$\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2h_2 = \alpha p_1 \qquad (2.10)$$

$$\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_3h_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2 \qquad (2.11)$$

$$-\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2h_2 = \alpha q_1 \qquad (2.12)$$

$$\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]\left(2a_2^2h_2^2-a_3h_3\right) = \alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2 \qquad (2.13)$$

From (2.10) and (2.12) , we obtain

$$p_1=-q_1 \qquad (2.14)$$

and

$$2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2 a_2^2h_2^2=\alpha^2\left(p_1^2+q_1^2\right) \qquad (2.15)$$

Now from (2.11) , (2.13) and (2.15) , we obtain

$$2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_2^2h_2^2 = \alpha\left(p_2+q_2\right)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right) = \alpha\left(p_2+q_2\right)+\frac{\alpha(\alpha-1)}{2}\frac{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2^2h_2^2}{\alpha^2}.$$

Therefore we have

$$h_2^2\,a_2^2=\frac{\alpha^2\left(p_2+q_2\right)}{4^n(1+\lambda)^2+\alpha\left[2.3^n\left(1+2\lambda\right)-4^n(1+\lambda)^2\right]}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2|\le\frac{2\alpha}{h_2\sqrt{4^n(1+\lambda)^2+\alpha\left(2.3^n\left(1+2\lambda\right)-4^n(1+\lambda)^2\right)}}$$

Next, in order to find the bound on $|a_3|$ by subtracting (2.13) and (2.11), we obtain

$$2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]\left[a_3h_3-a_2^2h_2^2\right]$$
$$=\alpha\left(p_2-q_2\right)+\frac{\alpha\left(\alpha-1\right)}{2}\left(p_1^2-q_1^2\right)$$

$$2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_3h_3$$
$$=\alpha\left(p_2-q_2\right)+\frac{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]\alpha^2\left(p_1^2+q_1^2\right)}{2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}.$$

$$a_3h_3=\frac{\alpha\left(p_2-q_2\right)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}+\frac{\alpha^2\left(p_1^2+q_1^2\right)}{2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}.$$

Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$|a_3|\leq\frac{1}{h_3}\left(\frac{2\alpha}{(1-\lambda)3^n+\lambda 3^{n+1}}+\frac{4\alpha^2}{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}\right).$$

This completes the proof of Theorem 2.1.

III.COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $H_\Sigma(n,\beta,\lambda)$

Definition 3.1. A function $f(z)$ given by (1.1) is said to in the class $H_\Sigma(n,\beta,\lambda)$ if following conditions are satisfied:

$$f\in\Sigma \ \ and$$
$$\operatorname{Re}\left\{\frac{(1-\lambda)D^n(f*h)(z)+\lambda D^{n+1}(f*h)(z)}{z}\right\}>\beta$$
$$;\left(0<\beta\leq 1,\lambda\geq 1,\ n\in N_0,z\in U\right)\qquad(3.1)$$

and

$$f\in\Sigma \ \ and$$
$$\operatorname{Re}\left\{\frac{(1-\lambda)D^n(f*h)^{-1}(w)+\lambda D^{n+1}(f*h)^{-1}(w)}{w}\right\}>\beta$$
$$;\left(0<\beta\leq 1,\lambda\geq 1,\ n\in N_0,z\in U\right)\qquad(3.2)$$

Where the function g is defined by (2.3).

We note that for $n=0$ and $\lambda=1$, the class $H_\Sigma(n,\beta,\lambda)$ reduce to the classes $H_\Sigma(\beta,\lambda)$ and $H_\Sigma(\lambda)$ studied by Frasin and Aouf [11] and Srivastava et al. [7], respectively.

Theorem 3.1. Let the function $f(z)$ given by (1.1) be in the class $H_\Sigma(n,\beta,\lambda)$, $n\in N_0$, $0<\beta\leq 1$ and $\lambda\geq 1$.

Then
$$|a_2|\leq\frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(1-\lambda)3^n+\lambda 3^{n+1}}}\qquad(3.3)$$

and
$$|a_3| \leq \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2} + \frac{2(1-\beta)}{\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}\right) \quad (3.4)$$

Proof: It follows from (3.1) and (3.2) that there exists $p(z)\in P$ and $q(z)\in P$ such that

$$\frac{(1-\lambda)D^n(f*h)(z)+\lambda D^{n+1}(f*h)(z)}{z} = \beta+(1-\beta)p(z) \quad (3.5)$$

and

$$\frac{(1-\lambda)D^n(f*h)^{-1}(w)+\lambda D^{n+1}(f*h)^{-1}(w)}{w} = \beta+(1-\beta)q(w) \quad (3.6)$$

Where $p(w)$ and $q(w)$ have the form (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2h_2 = (1-\beta)p_1 \quad (3.7)$$

$$\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_3h_3 = (1-\beta)p_2 \quad (3.8)$$

$$-\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]a_2h_2 = (1-\beta)q_1 \quad (3.9)$$

and
$$\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]\left(2a_2^2h_2^2-a_3h_3\right) = (1-\beta)q_2 \quad (3.10)$$

$$p_1=-q_1 \quad (3.11)$$

From (3.7) and (3.9) , we have

and
$$2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2 a_2^2h_2^2 = (1-\beta)^2\left(p_1^2+q_1^2\right) \quad (3.12)$$

Now from (3.8) , (2.13) and (3.10) , we find that

$$2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]a_2^2h_2^2 = (1-\beta)(p_2+q_2) \quad (3.13)$$

$$h_2^2 a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}$$

$$|a_2^2| \leq \frac{(1-\beta)(|p_2|+|q_2|)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]h_2^2} = \frac{(1-\beta)}{\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]h_2^2}$$

Which is the bound on $|a_2|$ as given in (3.3) .

Next, in order to find the bound on $|a_3|$ by subtracting (3.10) from (3.8), we obtain

$$2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]\left[a_3h_3-a_2^2h_2^2\right] = (1-\beta)(p_2-q_2)$$

or, equivalently

$$a_3 = \frac{1}{h_3}\left\{a_2^2h_2^2+\frac{(1-\beta)(p_2-q_2)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}\right\}.$$

Upon substituting the value of a_2^2 from (3.12), we have

$$\mathrm{a}_3 = \frac{1}{h_3}\left(\frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]}+\frac{(1-\beta)\left(p_2-q_2\right)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}\right).$$ Applying Lemma 1.1 once again for the coefficients p_1, p_2, q_1 and q_2, we obtain

$$\mathrm{a}_3 = \frac{1}{h_3}\left(\frac{(1-\beta)^2\left(p_1^2+q_1^2\right)}{2\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}+\frac{(1-\beta)\left(p_2-q_2\right)}{2\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}\right). \qquad \left|\mathrm{a}_3\right| \le \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{\left[(1-\lambda)2^n+\lambda 2^{n+1}\right]^2}+\frac{2(1-\beta)}{\left[(1-\lambda)3^n+\lambda 3^{n+1}\right]}\right),$$

Which is the bound on $|a_3|$ as given in (3.4).

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