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Some Curvature properties on Trans-Sasakian Manifolds

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ABSTRACT

In this paper we show that the trans-Sasakian manifolds satisfying the condition $R(X, Y)\cdot S = 0$ is an Einstein manifold. Finally, we show that $R\cdot\tilde{C} = R\cdot R$

Keywords: Trans-Sasakian Manifolds, Concircular Curvature Tensor, Einstein.

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Introduction

A class of almost contact metric manifolds known as trans-Sasakian manifolds was introduced by J.A.Oubina [6] in 1985. This class contains α-Sasakian, β-Kenmotsu and co-symplectic manifolds.

Trans-Sasakian manifolds are an important generalization of Sasakian, Kenmotsu and co-symplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. An almost contact metric structure on a manifold M is called a trans-Sasakian structure if the product manifold $M\times R$ belongs to the class W_4, a class of Hermitian manifolds which are closely related to a locally conformal Kahler manifolds. Trans-Sasakian manifolds were studied extensively by J.C. Marrero [5], C.S. Bagewadi and Venkatesha [1, 2], M.M. Tripathi [9] and others. Trans-Sasakian manifolds are an important generalization of Sasakian and cosymplectic manifolds in differential geometry. They arise naturally in the study of contact geometry and Riemannian geometry. Trans-Sasakian manifolds are used in theoretical physics, particularly in string theory and contact mechanics. They also appear in Hamiltonian dynamics, differential geometry, and sub-Riemannian geometry. They provide a unifying framework to study different geometric structures that arise naturally in complex geometry and topology.

In this paper, we study the trans-Sasakian manifolds satisfying the conditions $R(X, Y)\cdot S = 0$, is Einstein. Finally, we show that $R\cdot\tilde{C} = R\cdot R$, where $\tilde{C}$ is concircular curvature tensor.

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Preliminaries

An n-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits a $(1, 1)$ tensor field φ, a vector field ξ, a 1-form η and a Riemannian metric g, which satisfy

(2.1) $$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0, \quad \eta(\xi) = 1,$$

(2.2) $$g(\varphi X, Y) = -g(X, \varphi Y), \quad \eta(X) = g(X, \xi),$$

(2.3) $$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on M.

An almost contact metric manifold $M(\varphi, \xi, \eta, g)$ is said to be trans-Sasakian manifold if $(M \times R, J, G)$ belongs to the class W_4 of the Hermitian manifolds, where J is the almost complex structure on $M \times \boldsymbol{R}$ defined for any vector field Z on M and smooth function f on $M \times \boldsymbol{R}$ and G is the product metric on $M \times R$. This may be stated by the condition

(2.4) $$(\nabla_X\varphi) Y = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\varphi X, Y)\xi - \eta(Y)\varphi X\},$$

where α, β are smooth functions on M and such a structure is said to be the trans-Sasakian structure of type (α, β). From (2.4) it follows that

(2.5) $$\nabla_X\xi = -\alpha\varphi X + \beta\{X - \eta(X)\xi\}.$$

Note:

(1) If we consider α and β are smooth functions in equation (2.4) and $\alpha \neq 0$, $\beta = 0$ then the trans-Sasakian manifolds of type (α, β) reduces as α-Sasakian manifolds. Similarly, if α and β are scalars and $\alpha = 1$, $\beta = 0$ then the trans-Sasakian manifolds reduces as Sasakian manifolds.

(2) If we consider α and β are smooth functions in equation (2.4) and $\alpha = 0$, $\beta \neq 0$ then the trans-Sasakian manifolds of type (α, β) reduces as β-Kenmotsu manifolds. Similarly, if α and β are scalars and $\alpha = 0$, $\beta = 1$ then the trans-Sasakian manifolds reduces as Kenmotsu manifolds.

In a trans-Sasakian manifold $M(\varphi, \xi, \eta, g)$ the following relations hold:

(2.6) $$R(X, Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - (X\alpha)\varphi Y - (X\beta)\varphi^2 Y + 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X + (Y\beta)\varphi^2 X,$$

(2.7) $$\eta(R(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - 2\alpha\beta[g(\varphi X, Z)\eta(Y) - g(\varphi Y, Z)\eta(X)] - (Y\alpha)g(\varphi X, Z) - (X\beta)\{g(Y, Z) - \eta(Y)\eta(Z)\} + (X\alpha)g(\varphi Y, Z) + (Y\beta)\{g(X, Z) - \eta(X)\eta(Z)\},$$

(2.8) $$R(\xi, X)\xi = (\alpha^2 - \beta^2 - (\xi\beta))[\eta(X)\xi - X],$$

(2.9) $$S(X,\ \xi) = [(n-1)(\alpha^2-\beta^2)-(\xi\beta)]\eta(X)-((\varphi X)\alpha)-(n-2)(X\beta),$$

(2.10) $$S(\xi,\ \xi) = [(n-1)(\alpha^2-\beta^2)-(\xi\beta)],$$

(2.11) $$\xi\alpha + 2\alpha\beta = 0.$$

where R is the curvature tensor of type $(1, 3)$ and Q is the symmetric endomorphism of the tangent space at each point of the manifolds corresponding to the Ricci tensor S, that is, $g(QX,\ Y) = S(X,\ Y)$ for any vector fields $X,\ Y$ on M.

Lemma 2.1. *In a trans-Sasakian manifold of type (α, β), if*

(2.15) $$\varphi(grad\alpha) = (n-2)(grad\beta), \text{ then we have}$$

(2.16) $$\xi\beta = 0.$$

Thus the directional derivative of β with respect to characteristic vector field ξ is zero.

The concircular curvature tensor $\tilde{C}$ and Weyl projective tensor P on Trans-Sasakian manifold $M$of dimensional n is defined by

(2.18) $$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y],$$

for any vector fields X, Y, Z where R is the curvature tensor and r is the scalar curvature

Trans-Sasakian manifolds satisfying $R(X,\ Y)\cdot S = 0$

Definition 3.1. *An n-dimensional trans-Sasakian manifold M is said to be Ricci semi-symmetric if*

(3.1) $$R(X,\ Y)\cdot S = 0,$$

for any vecotr fields X, Y where R is the curvature tensor and S is the Ricci tensor

Theorem 3.1. *Let M be an n-dimensional trans-Sasakian manifold. Then M is Ricci-semi-symmetric if and only if an Einstein manifold.*

Proof. We know that every Einstein manifold is Ricci-semi-symmetric but the converse is not true in general. Here, we prove that in a trans-Sasakian manifolds $R(X,\ Y)\cdot S = 0$ implies that the manifold is an Einstein manifold.

(3.2) $$S(R(X,\ Y)U,\ V) + S(U,\ R(X,\ Y)V) = 0,$$

putting $X = \xi$ in equation (3.2), we have

(3.3) $$S(R(\xi,\ Y)U,\ V) + S(U,\ R(\xi,\ Y)V) = 0.$$

By using (2.6) in (3.3), we obtain

$$\begin{aligned}
&(\alpha^2-\beta^2)[g(Y,\ U)S(\xi,\ V) - \eta(U)S(Y,\ V) + g(Y,\ V)S(U,\ \xi) - \eta(V)S(U,\ Y)]\\
&+2\alpha\beta[g(\varphi U,\ Y)S(\xi,\ V) + \eta(U)S(\varphi Y,\ V) + g(\varphi V,\ Y)S(U,\ \xi) + \eta(V)S(U,\ \varphi Y)]\\
&+(U\alpha)S(\varphi Y,\ V) + g(\varphi U,\ Y)S(grad\alpha,\ V) + (U\beta)[S(Y,\ V) - \eta(Y)S(\xi,\ V)]\\
&-g(\varphi U,\ \varphi Y)S(grad\beta,\ V) + (V\alpha)S(U,\ \varphi Y) + g(\varphi V,\ Y)S(U,\ grad\alpha)
\end{aligned}$$

(3.4) $$+(V\beta)[S(U,\ Y) - \eta(Y)S(U,\ \xi)] - g(\varphi V,\ \varphi Y)S(U,\ grad\beta) = 0.$$

By putting $U = \xi$ in (3.4) and by using (2.9), (2.10), (2.11) and (2.16), we obtain

(3.5) $$S(Y,\ V) = (n-1)(\alpha^2-\beta^2)g(Y,\ V).$$

Therefore, M is Einstein manifold. This completes the proof of the theorem.

Now we consider $R \cdot \tilde{C}$ and prove the following theorem:

Theorem 3.2. *Let M be an n-dimensional trans-Sasakian manifold. Then*

$$R \cdot \tilde{C} = R \cdot R. \tag{3.6}$$

Proof. We have

$$\begin{aligned}(R(X,Y)\cdot \tilde{C})(U,V,W) &= R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W \\ &\quad -\tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W.\end{aligned} \tag{3.7}$$

In view of (2.18) in (3.7), we have

$$\begin{aligned}(R(X,Y)\cdot \tilde{C})(U,V,W) &= R(X,Y)\{R(U,V)W - \frac{r}{n(n-1)}[g(V,W)U - g(U,W)V]\} \\ &\quad - \{R(R(X,Y)U,V)W - \frac{r}{n(n-1)}[g(V,W)R(X,Y)U - g(R(X,Y)U,W)V]\} \\ &\quad - \{R(U,R(X,Y)V)W - \frac{r}{n(n-1)}[g(R(X,Y)V,W)U - g(U,W)R(X,Y)V]\} \\ &\quad - \{R(U,V)R(X,Y)W - \frac{r}{n(n-1)}[g(V,R(X,Y)W)U - g(U,R(X,Y)W)V]\}.\end{aligned} \tag{3.8}$$

We have

$$\begin{aligned}(R(X,Y)\cdot \tilde{C})(U,V,W) &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W \\ &\quad -R(U,V)R(X,Y)W + \frac{r}{n(n-1)}[g(V,R(X,Y)W)U - g(U,R(X,Y)W)V \\ &\quad -g(R(X,Y)U,W)V + g(R(X,Y)V,W)U].\end{aligned} \tag{3.9}$$

Finally, we get

$$(R(X,Y)\cdot \tilde{C})(U,V,W) = (R(X,Y)\cdot R)(U,V,W). \tag{3.10}$$

Therefore, $R \cdot \tilde{C} = R \cdot R.$ This completes the proof of the theorem.

From the above theorem we conclude that if Riemannian manifold is semi-symmetric satisfy $R(X, Y)\cdot R =$ 0 in (3.10) then the condition for concircularly semi-symmetric that is $R(X, Y)\cdot \tilde{C} = 0$ also satisfies in trans-Sasakian manifolds. From this we state the following corollary:

Corollary 3.3. *A trans-Sasakian manifold M is Concircularly semi-symmetric if and only if it is semi-symmetric Riemannian manifold.*

Corollary 3.4. [1] *If in a trans-Sasakian manifold M the relation* $R(X, Y) \cdot \tilde{C} = 0$ *holds then the manifold is concircularly flat.*

In general, a concircularly flat Riemannian manifold is Einstein and so, in particular, a concircu- larly flat trans-Sasakian manifold is Einstein. Hence we can state:

Corollary 3.5. [1] *A trans-Sasakian manifold M satisfying* $R(X, Y) \cdot \tilde{C} = 0$ *is an Einstein manifold.*

Conclusion

In a trans-Sasakian manifold if $R(X, Y) \cdot S = 0$ then the manifold is Einstein manifold. Further, it is shown that $R \cdot \tilde{C} = R \cdot R$ in trans-Sasakian manifold which implies $R(X, Y) \cdot R = 0$ if and only if $R(X, Y) \cdot \tilde{C} = 0$ that is the manifold is concircularly flat. Trans-Sasakian manifolds serve as a bridge between Sasakian, Kenmotsu and cosymplectic geometries, making them a rich area of study in modern differential geometry. Researchers continue to explore their curvature properties, classification, and applications in various fields of mathematics and physics. The concircular curvature tensor provides a refined way to measure the deviation of a manifold from constant curvature while preserving geodesic concircularity. It is particularly useful in trans-Sasakian geometry, Einstein manifolds, and conformal geometry. Understanding its properties allows for deeper insights into the geometric and physical interpretations of various manifolds.

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