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Connectivity of The Mycielskian of A Graph

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INTRODUCTION

We use V in place of $V(G)$, and E in place of $E(G)$ when no ambiguity arises. Moreover, for $S \subset V(G)$, $G \setminus S$ denotes the subgraph of G induced by the vertices of $V(G) \setminus S$. Similarly, for a vertex u of G, $S-u$ means $S \setminus \{u\}$.

The *connectivity* $\kappa(G)$ of a connected graph G is the least positive integer k such that there exists $S \subset V(G)$, $|S| = k$ and $G \setminus S$ is disconnected or reduces to the trivial graph K_1. An obvious inference from the definition of $\mu(G)$ is that $d_{\mu(G)}(x') = d_G(x)+1$ for all $x \in V(G)$. Consequently, $\delta(\mu(G)) = \delta(G)+1$ (here d stands for the degree and δ for the minimum degree). Also $\chi(\mu(G)) = \chi(G)+1$. Chang et.al., have proved Lemma 3.1.1.

LEMMA 1. If G has no isolated vertices, then $\kappa(\mu(G)) \geq \kappa(G)+1$.

PROOF. Suppose $V(G) = V$ and $V(\mu(G)) = V \cup V' \cup \{u\}$.

Let S be a subset of $V(\mu(G))$ of size $\kappa(G)$.

If $|S \cap V| < \kappa(G)$, then $G \setminus (S \cap V)$ is connected.

Also, for any vertex $x \in V$, x' is adjacent to at least $\kappa(G)$ vertices of V in $\mu(G)$. So, any such vertex x' of $\mu(G) \setminus S$ is adjacent to at least one vertex in $G \setminus (S \cap V)$. And u is adjacent to all such vertices x' of $\mu(G) \setminus S$. Thus, $\mu(G) \setminus S$ is connected.

If $|S \cap V| < \kappa(G)$, then $S \subseteq V$. Since G has no isolated vertices, any vertex $x \in V \setminus S$ is adjacent to some vertex y' in V', which is in turn adjacent to u. Thus, $\mu(G) \setminus S$ is also connected.

Therefore, $\kappa(\mu(G)) \geq \kappa(G)+1$. This completes the proof.

CONNECTIVITY OF THE MYCIELSKIAN

We first study a necessary and sufficient condition for $\kappa(\mu(G)) \geq \kappa(G)+1$ and see that this result is used to obtain a characterization for $\kappa(\mu(G))$. We make use of the following Remark 1 and Theorem 1 to prove our main result on connectivity.

REMARK 1. If S is a minimum vertex cut of G with $|S| = \kappa(G)$ and S' is the corresponding set of twins in V', then $S \cup S' \cup \{u\}$ is a vertex cut off $\mu(G)$. Therefore, $\kappa(G)+1 \leq \kappa(\mu(G)) \leq 2\kappa(G)+1$.

THEOREM 1. For a connected graph G, $\kappa(\mu(G)) = \kappa(G)+1$ if and only if $\delta(G) = \kappa(G)$.

PROOF. Let $\delta(G) = \kappa(G)$. Then $\kappa(\mu(G)) \leq \delta(\mu(G)) = \delta(G)+1 = \kappa(G)+1$.

Further by Lemma 1, $\kappa(\mu(G)) \geq \kappa(G)+1$.

Therefore, $\kappa(\mu(G)) = \kappa(G)+1$.

Conversely, let $\kappa(\mu(G)) = \kappa(G)+1$.

Suppose $\delta(G) \neq \kappa(G)$, then $1 \leq \kappa(G) < \delta(G)$.

Let $S = \{w_1, w_2, \ldots, w_{\kappa(G)+1}\}$ be a minimum vertex cut of $\mu(G)$.

Case a: $u \notin S$. Suppose $|V \cap S| \geq \kappa(G)$, then $|V \cap S| \geq \kappa(G)+i$, $i=0$ or 1 and there is a possibility for G to get disconnected. But since $\delta(G) \geq 2$, every vertex in $G \setminus (S \cap V)$ is adjacent to at least two vertices in V' which in turn are adjacent to u.

Hence, even if we remove an additional vertex from V' the resulting graph will remain connected, that is, $\mu(G) \setminus S$ is connected, a contradiction to the fact that S is a vertex cut.

If $|V \cap S| < \kappa(G)$, then $G \setminus (S \cap V)$ is connected and every vertex $x' \in V'$ is adjacent to at least $\kappa(G)+1$ vertices of G and hence adjacent to at least one vertex of $G \setminus (S \cap V)$. Also u is adjacent to all x' s in V'.

Thus, $\mu(G) \setminus S$ is connected, again contradicting the fact that S is vertex cut.

Case b: $u \in S$. Now remove u from $\mu(G)$ and set $G' = \mu(G) - u$. G' is connected (as $|S| \geq 2$).

To disconnect G' we have to remove the remaining $\kappa(G)$ vertices of S. Since $\delta(G) > \kappa(G)$, every vertex in G' is of degree at least $\kappa(G)+1$.

If $|V \cap (S-u)| < \kappa(G)$, then $G \setminus (V \cap (S-u))$ is connected and every vertex x' in V' is adjacent to at least $\kappa(G)+1$ vertices of G and hence to at least one vertex of $G \setminus (V \cap (S-u))$, so that G'/S is connected, a contradiction.

If $|V \cap (S-u)| = \kappa(G)$, there is a possibility for $G \setminus (V \cap (S-u))$ to get disconnected. If $G \setminus (V \cap (S-u))$ is connected, we get a contradiction as in *case a*. So let $G \setminus (V \cap (S-u))$ be disconnected and $G_1, G_2, \ldots, G_k$ its components.

Since every vertex of $V \cap (S-u)$ is adjacent to all the components $G_1, G_2, \ldots, G_k$, the twins of $V \cap (S-u)$ will be adjacent to all the components, that is $G \setminus (V \cap (S-u))$ together with the twins of $V \cap (S-u)$ is connected and each x' in V' is adjacent to at least one vertex of $G \setminus (V \cap (S-u))$. Therefore, $\mu(G) \setminus S$ is connected, which is again a contradiction.

Thus, $\delta(G)=\kappa(G)$. This completes the proof.

COROLLARY 1. If G is a connected graph, then $\kappa(\mu(G))=\kappa(G)+n$ if and only if $\delta(G)=\kappa(G)$.

MAIN RESULT ON CONNECTIVITY OF MYCIELSKI GRAPH

THEOREM 2. If G is a connected graph, then $\kappa(\mu(G))=\kappa(G)+i+1$ if and only if $\delta(G)=\kappa(G)+i$ for each $i,\ 0\le i\le\kappa(G)$.

PROOF. By induction on i, Theorem 1 gives the case when $i=0$.

So, assume that the result is true for all $j\le i-1$, that is $\kappa(\mu(G))=\kappa(G)+j+1$ if and only if $\delta(G)=\kappa(G)+j,\ j\le i-1$.

We now prove the result for $i(<\kappa(G))$. First consider the case when $\delta(G)=\kappa(G)+i$.

We know that $\kappa(\mu(G))\le\delta(\mu(G))=\delta(G)+1=\kappa(G)+i+1$. If $\kappa(\mu(G))<\kappa(G)+i$, by induction hypothesis, $\delta(G)<\kappa(G)+i$. Therefore $\kappa(\mu(G))=\kappa(G)+i+1$.

Conversely, let $\kappa(\mu(G))=\kappa(G)+i+1$.

Suppose, then $\delta(G)\ne\kappa(G)+i$, then $\delta(G)>\kappa(G)+i$ (because if $\delta(G)=\kappa(G)+i$, then by induction hypothesis $\kappa(\mu(G))\ne\kappa(G)+i+1$). Let $S=\{w_1,w_2,\dots,w_{\kappa(G)+i+1}\}$ be a minimum vertex cut of $\mu(G)$.

Case a: $u\notin S$. Suppose $|S\cap V|\ge\kappa(G)$, then $|S\cap V|=\kappa(G)+l,\ 0\le l\le i+1$. Every vertex $x\in V\setminus(V\cap S)$ is adjacent to at least $\delta(G)$ vertices of G and hence (by the definition of Mycielskian) to at least $\kappa(G)+i+1$ vertices in V' and hence to at least one vertex in $V'\setminus(S-(V\cap S))$ which in turn is adjacent to u.

Therefore, $\mu(G)\setminus S$ is connected, which is a contradiction to the fact that S is a vertex cut of $\mu(G)$.

Suppose now $|S\cap V|<\kappa(G)$. Then $G\setminus(V\cap S)$ is connected and every vertex $x'\in V'$ is adjacent to at least $\kappa(G)+i+1$ vertices of G and hence adjacent to at least one vertex of $G\setminus(V\cap S)$. Also, u is adjacent to all such vertices x'.

Therefore, $\mu(G)\setminus S$ is connected, a contradiction.

Case b: $u\in S$. As before, set $G'=\mu(G)-u$. G' is connected.

To disconnect G' we have to remove all the remaining $\kappa(G)+i$ vertices of S. Since $\delta(G)>\kappa(G)+i$, every vertex in G' is of degree at least $\kappa(G)+i+1$.

If $|V\cap(S-u)|<\kappa(G)$, then $G\setminus(V\cap(S-u))$ is connected and every vertex x' is adjacent to at least $\kappa(G)+i+1$ vertices of G and hence to at least one vertex of $G\setminus(V\cap(S-u))$.

Therefore, $\mu(G)\setminus S$ is connected which is not true. If $|V\cap(S-u)|=\kappa(G)+l,\ 0\le l\le i$, there is a possibility for $G\setminus(V\cap(S-u))$ to get disconnected. If $G\setminus(V\cap(S-u))$ is connected we get a contradiction as before.

So let $G\setminus(V\cap(S-u))$ be disconnected with components $G_1,G_2,\dots,G_k$. Since $V\cap(S-u)$ is a vertex cut of G, there will be at least $\kappa(G)$ vertices in $V\cap(S-u)$ that will be adjacent to all the components $G_1,G_2,\dots,G_k$ of G, call this set as T.

By the definition of Mycielskian, the twins of the set T in V' will be adjacent to all the components $G_1, G_2, \ldots, G_k$, that is, $G \setminus (V \cap (S-u))$ together with any of the twins of T is connected and since the number of vertices still to be removed is $i-l < \kappa(G) \leq |T|$, even after the removal of the whole set S there will be at least one twin, say, z' in $G' \setminus (S-u)$, of a vertex z in T.

Also, each x' s of V' is adjacent to at least one vertex of $G' \setminus (V \cap (S-u))$. Thus, $\mu(G) \setminus S$ is connected which again contradicts the fact that S is a vertex cut. Hence $\delta(G) = \kappa(G) + i$. This completes the proof.

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