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Ghaph Valued Functions

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ABSTRACT

In this paper, we obtain some basic results on minimal dominating graph, in particular a characterization of $MD(G)$ which are complete, eulerian, hamiltonian and connectedness. In addition, we find the relationship of $MD(G)$ with other graph valued functions.

Keywords : Minimal Dominating Graph, Domatic Number, Graph Valued Functions

1. INTRODUCTION

Let $G = (V, E)$ be a graph. A set $D \subseteq V$is called a dominating set if every vertex $v \in V$ is either an element of D or is adjacent to an element of D.

A dominating set D is a minimal dominating set if no proper subset $D' \subset D$ is a dominating set. The domination number $\gamma(G)$ of G is the minimum cardinality of a minimal dominating set in G. The upper domination number $\Gamma(G)$ of G is the maximum cardinality of a minimal dominating set in G.

Illustration:

The dominating sets of G are {2}, {1,2}, {1,3}, {1,4}, {2,3}, {2,4}, {1,2,3}, {1,3,4}, {1,2,4}. The minimal dominating sets of G are {2}, {1,4}, {1,3}.

$$\gamma(G) = 1, \Gamma(G) = 2$$

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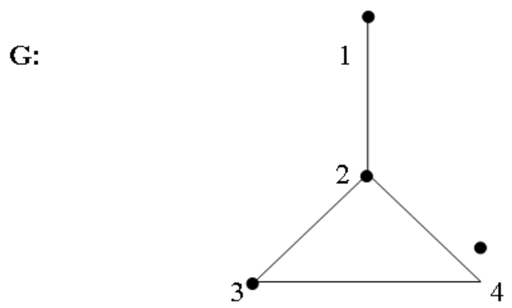
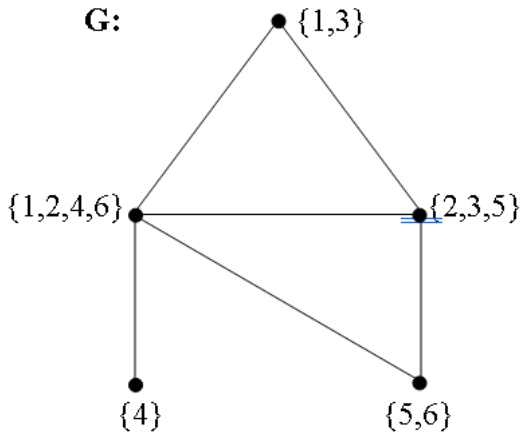


Let S be a finite set and $F = \{S_1, S_2, \ldots, S_n\}$ be a partition of S. Then the intersection graph $\Omega(F)$ of F is the graph whose vertices are the subsets in F and in which two vertices S_i and S_j are adjacent if and only if $S_i \cap S_j \neq \varphi$, for $i \neq j$.

Illustration:

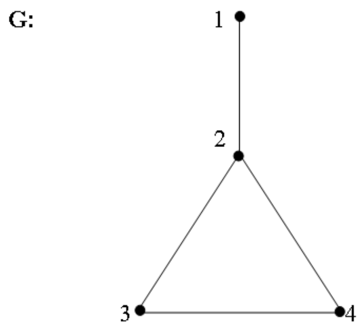


$S = \{1, 2, 3, 4, 5, 6\}$

$F = \{\{4\}, \{1,3\}, \{5,6\}, \{1,2,4,6\}, \{2,3,5\}\}$

Domatic number $d(G)$ of a graph G to be the largest order of a partition of $V(G)$ into dominating set of G.

Illustration:



$P_1 = \{\{2\}, \{1,3,4\}\}$

$P_2 = \{\{2,4\}, \{1,3\}\}$

$P_3 = \{\{3,2\}, \{1,4\}\}$

$d(G) = 2$

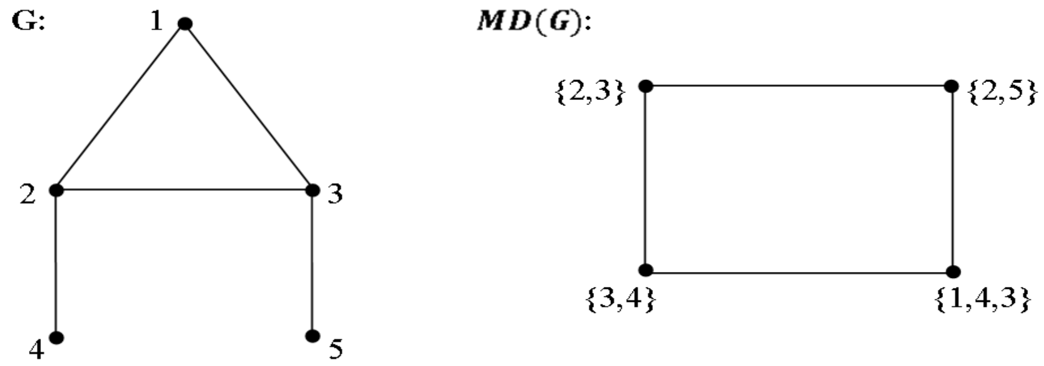
A clique in a graph G is a maximal complete subgraph of G. The order of the largest clique in a graph G is its clique number, which is denoted by $\omega(G)$. (The symbol ω is the Greek letter "Omega").

The clique graph of a given graph G is the intersection graph of cliques of G and it is denoted by $K(G)$.

The line graph $L(G)$ of a graph G is the graph whose vertex set corresponds to the edges of G such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.The independence graph $I(G)$ of a graph G is defined to be the intersection graph on the independent sets of vertices of G (see [2]).

The independence domination number *i(G)* of G is the minimum cardinality among all independent dominating sets of G.

DEFINITION: The minimal dominating graph $MD(G)$ of a graph G is the intersection graph defined on the family of all minimal dominating sets of vertices of G. We illustrate this concept through Fig 2.1.



A graph and its minimal dominating graph

Figure. 2.1

The following Theorems are useful to prove our next results.

THEOREM 2.A [5]. Every maximal independent set in a graph G is a minimal dominating set of G.

THEOREM 2.B [3]. If G is a graph without isolated vertices and S is a minimal dominating set of G then $V(G) - S$ is a dominating set of G.

THEOREM 2.C [5]. For any graph $G, \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G)$

THEOREM 2.D [4]. A graph G is eulerian if and only if every vertex of G has even degree.

THEOREM 2.E [4]. If for each vertex v of G, $deg\ v \geq {}^{p}/_{2}$, where $p \geq 3$, then G is hamiltonian.

2. SOME BASIC PROPERTIES OF THE MINIMAL DOMINATING GRAPH

THEOREM 2.1. For any graph G with at least two vertices, $MD(G)$ is connected if and only if $\Delta(G) < p - 1$.

PROOF: Let $\Delta(G) < p - 1$. Let D_1 and D_2 be two disjoint minimal dominating sets of G. We consider the following cases:

Case 1. Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that u and v are not adjacent. Then there exists a maximal independent set D_3 containing u and v. Since by Theorem 2.A, D_3 is minimal dominating set, this implies that D_1 and D_2 are connected in $MD(G)$ through D_3.

Case 2. Suppose every vertex in D_1 is adjacent to every vertex in D_2. We consider the following two subcases:

Subcase 2.1. Suppose there exist two vertices $u \in D_1$ and $v \in D_2$ such that every vertex not in $D_1 \cup D_2$ is adjacent to either u or v. Then $\{u, v\}$ is a minimal dominating set of G. Hence D_1 and D_2 are connected in $MD(G)$ through $\{u, v\}$.

Subcase 2.2. Suppose for any two vertices $u \in D_1$ and $v \in D_2$, there exists a vertex $w \notin D_1 \cup D_2$ such that w is adjacent to neither u nor v. Then there exist two maximal independent sets D_3 and D_4 containing u, w and v respectively. Thus, as above D_1 and D_2 are connected in $MD(G)$ throughD_3 and D_4.

Thus, from the above cases every two vertices in $MD(G)$ are connected and hence $MD(G)$ is connected.

Conversely, suppose $MD(G)$ is connected. Assume $\Delta(G) = p - 1$ and u is a vertex of degree $p - 1$. Then $\{u\}$ is a minimal dominating set of G and $V - \{u\}$ also contains a minimum dominating set of G. This shows that $MD(G)$ has at least two components, a contradiction. Hence $\Delta(G) < p - 1$. This completes the proof.

We characterized graphs G whose minimal dominating graphs $MD(G)$ are complete.

THEOREM 2.2. The minimal dominating graph $MD(G)$ of a graph G is complete if and only if G contains isolated vertex.

PROOF: Let u be an isolated vertex of G. Then u is in every minimal dominating set of G. Hence every two vertices in $MD(G)$ are adjacent. Thus $MD(G)$ is complete. Conversely, suppose $MD(G)$ is complete. Assume G has no isolated vertex. Let D be a minimal dominating set of G. Then $V - D$ contains a minimal dominating set D'. Then D and D' are two non adjacent vertices in $MD(G)$, a contradiction. Which shows that G contains an isolated vertex. This completes the proof.

The following result gives the existence of the minimal dominating graph of a graph.

THEOREM 2.3. The minimal dominating graph $MD(G)$ of a graph G is either connected or has at most one component that is not K_1.

PROOF: We consider the following three cases:

Case 1. If $\Delta(G) < p - 1$,then by Theorem 2.1, $MD(G)$ is connected.

Case 2. If $\Delta(G) = \delta(G) = p - 1$, then $G = K_p$ and hence each $\{v\} \subseteq V$ is a minimum dominating set of G and hence each component of $MD(G)$ is K_1.

Case 3. Suppose $\delta(G) < \Delta(G) = p - 1$. Let $u_1, u_2, \dots, u_n$ be the vertices having degree $p - 1$. Let $H = G \backslash \{u_1, u_2, \dots, u_n\}$. Then clearly $\Delta(H) < V(H) - 1$ and hence by Theorem 2.1, $MD(G)$ is connected. Since $MD(G) = \Omega(V(MD(H)) \cup \{u_1\} \cup \{u_2\} \cup \dots \cup \{u_n\})$, then exactly one component of $MD(G)$ is not K_1. This completes the proof.

The next result relates to the independence number $\beta_0(MD(G))$ and domatic number $d(G)$.

THEOREM 2.4. For any graph G, $\beta_0(MD(G)) = d(G)$ ---------- (1)

PROOF: Let F be a maximum order domatic partition of $V(G)$. If each dominating set in F is minimal, then F is a maximum independent set of vertices in $MD(G)$ and hence $\beta_0(MD(G)) = d(G)$. Otherwise, let $D \subseteq F$ be a dominating set in F which is not minimal. Then there exists a minimal dominating set $D' \subset D$. Replacing each such D in F by its subset D', we see that F is a maximum independent set of vertices in $MD(G)$. Thus $\beta_0(MD(G)) = d(G)$.This completes proof.

COROLLARY 2.5. For any graph G,

$|V(MD(G))| \geq d(G)$ ---------- (2)

$\gamma(MD(G)) \leq \delta(G) + 1$ ---------- (3)

$\gamma(MD(G)) \leq p - \gamma(G) + 1$ ---------- (4)

where $\delta(G)$ is the minimum degree of G and $V(MD(G))$ and $\gamma(MD(G))$ are the vertex set and domination number of $MD(G)$, respectively.

PROOF: (2) follows from (1) and the fact that, for any graph G,

$$|V(G)| \geq \beta_0(G) \qquad \Rightarrow \qquad |V(MD(G))| \geq \beta_0(MD(G))$$

Also by (1) $\quad \beta_0(MD(G)) = d(G) \qquad \Rightarrow |V(MD(G))| \geq d(G).$

We prove (3), since by the Theorem 2.C

$$\gamma(G) \leq \beta_0(G) \qquad \Rightarrow \gamma(MD(G)) \leq \beta_0(MD(G))$$

Also, we know that by (1) $\quad \gamma(MD(G)) \leq d(G)$

In [1] Cockayne and Hedetniemi shown that

$$d(G) \leq \delta(G) + 1$$

$$\gamma(MD(G)) \leq \delta(G) + 1.$$

Finally, we prove (4). Since, $\gamma(G) \leq p - \delta(G)$

$$\gamma(MD(G)) \leq p - \gamma(G) + 1.$$

This completes the proof.

In the next result, they characterized graphs whose minimal dominating graphs have domination number equal to the order of G.

THEOREM 2.6. For any graph G, $\gamma(MD(G)) = p$ ---------- (5)

if and only if every independent set of G is a dominating set.

PROOF: Suppose every independent set of G is a dominating set. Then each $\{v\} \subseteq V$ is a minimal dominating set of G. This proves that $MD(G) = \overline{K_p}$. Hence (5) holds.

Conversely, suppose (5) holds. Then by (3)

$$\gamma(MD(G)) \leq \delta(G) + 1 \qquad \Rightarrow p \leq \delta(G) + 1$$

it follows that $\delta(G) = p - 1$. Hence $G = K_p$. Thus, every independent set of G is dominating set. This completes the proof.

They also characterized graphs whose minimal dominating graphs have domatic number one.

THEOREM 2.7. For any graph G, $d(MD(G)) = 1$ ... (6)

if and only if $G = \overline{K_p}$ or $\Delta(G) = p - 1$, where $\overline{K_p}$ is complement of K_p.

PROOF: Suppose (6) holds. Then $MD(G)$ contains an isolated vertex D. If $V(MD(G)) = \{D\}$, then $D = V$, and hence $G = \overline{K_p}$. Otherwise, $MD(G)$ is disconnected and hence by Theorem 2.1, $\Delta(G) = p - 1$. The converse is obvious. This completes the proof.

THEOREM 2.8. For any graph G, $\omega(G) \leq |V(MD(G))|$ ... (7)

where $\omega(G)$ is the clique number of G.

PROOF: Let S be a set of vertices in G such that the induced graph $\langle S \rangle$ is complete with $|S| = \omega(G)$. Then, for each vertex $v \in S$, there exists a minimal dominating set containing v and (7) follows. This completes the proof.

They characterized the graphs G for which $K(\overline{G}) = MD(G)$.

3. RELATIONSHIP WITH OTHER GRAPH VALUED FUNCTIONS

THEOREM 2.9. For any graph G, $K(\overline{G}) \subseteq MD(G)$ ... (8)

Furthermore, equality is attained if and only if every minimal dominating set of $G$is independent.

PROOF: Let S denote the family of all maximal independent sets of vertices in G. Then $\Omega(S) = K(\overline{G})$, and (8) follows from the fact that $\Omega(S) \subseteq MD(G)$. Therefore, $\quad K(\overline{G}) = \Omega(S) \subseteq MD(G)$

$$\Rightarrow K(\overline{G}) \subseteq MD(G)$$

Suppose the equality in (8) is attained. Then it follows that $MD(G) = \Omega(S)$. This prove that every minimal dominating set of G is independent.

Conversely, suppose every minimal dominating set D of G is independent. Then D is maximal independent set. Thus $MD(G) = \Omega(S)$ and hence the equality in (8) is attained. This completes the proof.

They also established the following other characterization.

COROLLARY 2.10. For any graph G, $L(\overline{G}) \subseteq MD(G)$ ... (9)
if and only if $\beta_0(G) \leq 2$.

PROOF: Suppose (9) holds. Then any two nonadjacent vertices of G form a minimal dominating set of G. This proves that $\beta_0(G) \leq 2$.

Conversely, suppose $\beta_0(G) \leq 2$. Then it follows that $\omega(\overline{G}) \leq 2$ and hence $L(\overline{G}) \subseteq K(\overline{G})$. Thus (9) follows from (8). This completes the proof.

COROLLARY 2.11. For any graph G, $MD(G) \subseteq I(G)$ ... (10)
if and only if every minimal dominating set of G is independent.

PROOF: Suppose every minimal dominating set of G is independent. Then by Theorem2.9, $K(\overline{G}) = MD(G)$. From Cockayne and Hedetniemi [2], $K(\overline{G}) \subseteq I(G)$. Hence (10) follows. The converse is immediate. This completes the proof.

4. EULERIAN AND HAMILTONIAN PROPERTIES OF MINIMAL DOMINATING GRAPH

They gave a sufficient condition on G for which the minimal dominating graph *MD(G)* is eulerian.

THEOREM 2.12. If G is a $(p-2)$ - regular graph, then $MD(G)$ is eulerian.

PROOF: Let $v \in V$ be a vertex in G. Then there exists exactly one vertex u such that u is not adjacent to v. This proves that $\{u, v\}$ is a minimal dominating set of G. Since for any vertex w adjacent to v, $\{v, w\}$ is also a minimal dominating set of G, for any minimal dominating set D of G, there exist exactly $2(p-2)$ minimal dominating sets containing a vertex of D. Thus, D has even degree in $MD(G)$ and hence by Theorem 2.D, $MD(G)$ is eulerian. This completes the proof.

The following result gives sufficient conditions on G for which the minimal dominating graph G is hamiltonian.

THEOREM 2.13. Let G be a graph satisfying the following conditions:

(i) For any two adjacent vertices u and v, $\{u, v\}$ is a minimal dominating set of G; and

(ii) $\delta(G) \geq \frac{1}{8}p(p-1)$.

Then, *MD(G)* is hamiltonian.

PROOF: Let $u \in V$ be a vertex of G. Then by (i), there exists a vertex w such that w is not adjacent to u. Let S be a maximal independent set containing u and w. S is also minimal dominating set and for any minimal dominating set D of G, there are atleast $2\delta(G)$ minimal dominating sets containing a vertex of D and hence every vertex in $MD(G)$ has degree at least $2\delta(G)$.

Also, if $\overline{q}$ denotes the number of edges in $\overline{G}$, then $|V(MD(G))| \leq q + \overline{q} = \frac{p(p-1)}{2} \leq 4\delta(G)$, using (ii). Thus, by Theorem 2.E, $MD(G)$ is hamiltonian. This completes the proof.

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