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On the Integer Solutions of the Pell Equation $x^2 - 79y^2 = 9^k$

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ABSTRACT

The binary quadratic Diophantine equation represented by $x^2 - 79y^2 = 9^k, \quad k > 0$ is considered. A method of obtaining infinitely many non-zero distinct integer solutions of the Pell equation considered above is illustrated. A few interesting relations among the solutions and some special numbers are presented and the recurrence relations on the solutions are obtained.

Keywords: Pell Equation, Binary Quadratic Diophantine Equation, Integer Solutions.

I. INTRODUCTION

Pell's equation is any Diophantine equation of the form $x^2 - ny^2 = 1$, when n is a given positive non-square integer has always positive integer solutions. This equation was first studied extensively in India, starting with Brahmagupta, who developed the Chakravala method to Pell's equation and other quadratic indeterminate equations. When k is a positive integer and $n \in (k^2 \pm 4, k^2 \pm 1)$, positive integer solutions of the equations $x^2 - ny^2 = \pm 4$ and $x^2 - ny^2 = \pm 1$, have been investigated by Jones in [2]. In [1],[4],[6],[7],[8], [9],[11] and [12] some special Pell equation and their solutions are considered. In [3], the integer solutions of the Pell equation $x^2 - dy^2 = 2^t$ has been considered. In [5], the Pell equation $x^2 - (k^2 - k)y^2 = 2^t$ is analyzed for its integer solutions. In [10], the Pell equation $x^2 - 3y^2 = (k^2 + 4k + 1)^t$ is analyzed for its positive integer solutions.

In this communication, we present the Pell equation $x^2 - 79y^2 = 9^k$, where $k > 0$ and infinitely many positive integer solutions are obtained. A few interesting relations among the solutions are presented. Further recurrence relations on the solutions are obtained.

Notations

$T_{m,n}$ = Polygonal number of rank n with sides m.

P_n^m = Pyramidal number of rank n with sides m.

G_n = Gnomonic number of rank n.

II. METHODS AND MATERIAL

Consider the Pell equation

$$x^2 - 79y^2 = 9^k \qquad (1)$$

Let (x_0, y_0) be the initial solution of (1) which is given by

$$x_0 = 80 \cdot 3^k, \ y_0 = 3^{k+2}, \qquad k \in z - \{0\}$$

To find the other solutions of (1), consider the Pellian equation

$$x^2 = 79y^2 + 1$$

whose general solution $(\tilde{x}_n, \tilde{y}_n)$ is given by

$$\tilde{x}_n = \frac{1}{2} f_n$$

$$\tilde{y}_n = \frac{1}{2\sqrt{79}} g_n$$

where

$$f_n = \left(80 + 9\sqrt{79}\right)^{n+1} + \left(80 - 9\sqrt{79}\right)^{n+1}$$
$$g_n = \left(80 + 9\sqrt{79}\right)^{n+1} - \left(80 - 9\sqrt{79}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the sequence of non-zero distinct integer solutions of (1) are obtained as

$$x_{n+1} = \frac{3^k}{2}\left[80 f_n + 9\sqrt{79} g_n\right]$$

$$y_{n+1} = \frac{3^k}{2\sqrt{79}}\left[9\sqrt{79} f_n + 80 g_n\right]$$

The recurrence relations satisfied by the solutions of (1) are given by

$$x_{n+1} - 160x_{n+2} + x_{n+3} = 0, \quad x_1 = 12799 \cdot 3^k, x_2 = 2047760 \cdot 3^k$$

$$y_{n+1} - 160y_{n+2} + y_{n+3} = 0, \quad y_1 = 1440 \cdot 3^k, y_2 = 230391 \cdot 3^k$$

III. RESULTS AND DISCUSSION

PROPERTIES

1. $x_{n+2} = 80x_{n+1} + 711y_{n+1}$
2. $y_{n+2} = 9x_{n+1} + 80y_{n+1}$
3. $24964(160x_{n+1} - 1422y_{n+1}) - 316(12640y_{n+1} - 1422x_{n+1})$ is a quadratic integer.
4. $160x_{2n+2} - 1422y_{2n+2} + 2 \cdot 3^k \equiv 0 \left(\bmod 3^k\right)$
5. $37446(160x_{n+1} - 1422y_{n+1}) - 474(12640y_{n+1} - 1422x_{n+1})$ is a nasty number.
6. When $k \equiv 0(\bmod 3)$, $160x_{3n+3} - 1422y_{3n+3} + 480x_{n+1} - 4266y_{n+1}$ is a cubic integer.
7. When $k \equiv 0(\bmod 4)$, $160x_{4n+4} - 1422y_{4n+4} + 4 \cdot 3^k T_{4,f_n} - 2 \cdot 3^k$ is a biquadratic integer.
8. $160x_{3n+3} - 1422y_{3n+3} - 2P_{f_n}^{\ 5} \cdot 3^k + 2T_{3,f_n} \cdot 3^k \equiv 0(\bmod f_n)$
9. $160x_{3n+3} - 1422y_{3n+3} - 6P_{f_n}^{\ 3} \cdot 3^k + G_{f_n} \cdot 3^k \equiv 0\left(\bmod 3^k\right)$

REMARKABLE OBSERVATIONS

By considering the linear combination among the solutions, one may obtain solutions of different hyperbolas. A few examples are given below

i) Define

SET 1:

$x = 160x_{n+1} - 1422y_{n+1}, \quad y = 12640y_{n+1} - 1422x_{n+1}$

SET 2:

$x = 320x_{n+1} - 2x_{n+2}, \quad y = 12640y_{n+1} - 1422x_{n+1}$

Note that in both sets (x, y) satisfies the hyperbola $79x^2 - y^2 = 2844.$

ii) Define

$x = 51198x_{n+1} - 2x_{n+3}, \quad y = 12640y_{n+1} - 1422x_{n+1}$

Note that (x, y) satisfies the hyperbola $79x^2 - 25600y^2 = 72806400.$

iii) Define

$x = 12640y_{n+2} - 2022242y_{n+1}, \ y = 12640y_{n+1} - 1422x_{n+1}$

Note that (x, y) satisfies the hyperbola $x^2 - 6399y^2 = 18198756$.

iv) Define

$x = 158y_{n+3} - 4044326y_{n+1}, \ y = 12640y_{n+1} - 1422x_{n+1}$

Note that (x, y) satisfies the hyperbola $x^2 - 25596y^2 = 72795024$.

v) Define

SET 1:

$x = 160x_{n+1} - 1422y_{n+1}, \quad y = 160x_{n+2} - 25598x_{n+1}$

SET 2 :

$x = 160x_{n+1} - 1422y_{n+1}, \ y = 25598x_{n+3} - 4095520x_{n+2}$

Note that in both sets (x, y) satisfies the hyperbola $6399x^2 - y^2 = 230364$.

vi) Define

$x = 160x_{n+1} - 1422y_{n+1}, \quad y = 2x_{n+3} - 51194x_{n+1}$

Note that (x, y) satisfies the hyperbola $25596x^2 - y^2 = 921456$.

IV. CONCLUSION

One may search for other patterns of solutions of the considered equation.

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