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A Study on Fixed Point Theorem of Brian Fisher and Others

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ABSTRACT

The aim of this paper is to prove a common fixed-point theorem which generalizes the result of Brian Fisher [1] and etal. by weaker conditions. The conditions compatibility and completeness of a metric space are replaced by weaker conditions such as and compatible mappings of type (p) and the Iterated sequence.
Keywords : Fixed Point, Self-Maps, Iterated Sequence, Compatible Maps, Compatible Mappings of Type (p).
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I. INTRODUCTION

Definition: Two self-maps A and S of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: Two self-maps A and S of a metric Space (X,d) are said to be compatible of type (P), if $\lim_{n\to\infty} d(AAx_n, SSx_n) = 0$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition: A and S are two self-maps of a metric space (X,d). In the pair (A,S), S is said to be continuous if $\lim_{n\to\infty} SSx_n = Sz$ and $\lim_{n\to\infty} SAx_n = Sz$ whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

II. METHODS AND MATERIAL

Literature for our main result

Brian Fisher and others proved the following common Fixed Point theorem for four self-maps of a complete metric space.

Theorem (A): Suppose A, B, S and T are four self-maps of metric space (X,d) such that

i. (X,d) is a complete metric space
ii. $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$
iii. The pairs(A,S)and,(B,T) are compatible
iv. $d(Ax,By)^2$

$\leq c_1 \max\{d(Sx,Ax)^2, d(Ty,By)^2, d(Sx,Ty)^2\} + c_2 \max\{d(Sx,Ax)\, d(Sx,By), d(Ax,Ty) d(By,Ty)\} + c_3\{d(Sx,By) d(Ty,Ax)\}$

Where $c_1, c_2, c_3 \geq 0$, $c_1 + 2c_2 < 1$ and $c_1 + c_3 < 1$, then A,B,S and T have a unique common fixed point $z \in X$.

We prove the existence of the result using weaker conditions such as and compatible mappings of type (p) and the iterated sequence. An example is given to justify the result.

Iterated Sequence : Suppose A, B, S and T are self-maps of a metric space(X,d) satisfying the condition (ii), then for any $x_0 \in X$, $Ax_0 \in A(X)$ and hence, $Ax_0 \in T(X)$ so that there is a $x_1 \in X$ with $Ax_0 = Tx_1$. Now $Bx_1 \in B(X)$ and hence there is $x_2 \in X$ with $Bx_1 = Sx_2$.Repeating this process to each $x0 \in X$, we get a sequence $<x_n>$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for $n \geq 0$. We shall call this sequence as an Iterated sequence of x_0 relative to the four self-maps A, B, S and T.

Now we prove a lemma, which plays an important role in proving our theorem.

Lemma : Suppose A, B, S and T are four self-maps of a metric space (X,d) satisfying the conditions (ii) and (iv) of Theorem (A) and Further if (L1): (X,d) is a complete metric space than for any $x_0 \in X$ and for any of its iterated sequence $<x_n>$ relative to four self-maps, the sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots$, converges to some point $z \in X$.

Proof: For simplicity let us take $d_n = d(y_n, y_{n+1})$ for $n=0,1,2,\ldots$

We have

$$d^2_{2n} = [d(y_{2n}, y_{2n+1})]^2$$

$$= [d(Ax_{2n}, Bx_{2n+1})]^2$$

$$\le c_1 \max\{[d(Sx_{2n}, Ax_{2n})]^2, d(Tx_{2n+1}, Bx_{2n+1})]^2, [d(Sx_{2n}, Tx_{2n+1})]^2\}$$
$$+ c_2 \max\{d(Sx_{2n}, Ax_{2n}) d(Sx_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n+1}) d(Bx_{2n+1}, Tx_{2n+1})\}$$
$$+ c_3 \max\{d(Sx_{2n}, Bx_{2n+1})\, d(Tx_{2n+1}, Ax_{2n})\}$$

$$= c_1 \max\{[d(y_{2n-1}, y_{2n})]^2, [d(y_{2n}, y_{2n+1})]^2, [d(y_{2n-1}, y_{2n})]^2\}$$
$$+ c_2 \max\{d(y_{2n-1}, y_{2n})\, d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n}) d(y_{2n+1}, y_{2n+1})\}$$
$$+ c_3 \max\{d(y_{2n-1}, y_{2n+1})\, d(y_{2n+1}, y_{2n})\}$$

$$= c_1 \max\{[d(y_{2n-1}, y_{2n})]^2, [d(y_{2n}, y_{2n+1})]^2, [d(y_{2n-1}, y_{2n})]^2\}$$
$$+ c_2 \max\{d(y_{2n-1}, y_{2n})\, d(y_{2n-1}, y_{2n+1})\}$$

$$\le c_1 \max\{d^2_{2n-1}, d^2_{2n}\} + c_2 \max\{d_{2n-1}[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})]\}$$

$$= c_1 \max\{d^2_{2n-1}, d^2_{2n}\} + c_2 \max[d^2_{2n-1} + d_{2n-1} d_{2n}]$$

$$\le c_1 \max\{d^2_{2n-1}, d^2_{2n}\} + c_2[\tfrac{3}{2} d^2_{2n-1} + \tfrac{1}{2} d^2_{2n}] \quad \ldots\ldots(*)$$

If $d_{2n} > d_{2n-1}$, inequality (*)

implies $0 < \left(\dfrac{3c_2}{2 - 2c_1 - c_2} - 1\right) d^2_{2n}$ a contradiction

since $\dfrac{3c_2}{2 - 2c_1 - c_2} < 1$.

Thus $d_{2n} \le d_{2n-1}$ and inequality (*) implies that $d_{2n} = d(y_{2n}, y_{2n+1}) \le h.d(y_{2n}, y_{2n+1}) = h.d_{2n-1}$

Where $h = \sqrt{\dfrac{2c_1 + 3c_2}{2 - c_2}} < 1$

Consequently, $d(y_n, y_{n+1}) \le h^n d(y_0, y_1)$ for $n = 1, 2, 3\ldots\ldots$

For every integer, $p > 0$,

we get $d(y_n, y_{n+p}) \le h^n d(y_0, y_1)$

$d(y_n, y_{n+p}) \le d(y_n, y_{n+1}) +$

$d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+p-1}, y_{n+p})$

$d(y_n, y_{n+p}) \le h^n(1 + h + h^2 + h^3 + \ldots . h^p) d(y_0, y_1)$

Since $0 \le h < 1$, $h^n \to 0$ as $n \to \infty$, so that $d(y_n, y_{n+p}) \to 0$. This shows that the sequence $\{y_n\}$ is a cauchy sequence in X and since X is a complete metric space it converges to a limit, say $z \in X$. Hence the Lemma.

The converse of the lemma is not true.

That is, suppose A, B, S and T are self-maps of a metric space (X, d) satisfying the conditions (ii) and (iv) of theorem(A), even for each iterated sequence $<x_n>$ of x_0, the sequence (L1) converges, the metric space (X,d) need not be complete. For this, we provide an example.

Example: Let $X = \left(0, \frac{1}{2}\right)$ with $d(x, y) = |x - y|$

$$A(x) = B(x) = \begin{cases} \frac{1}{4} & \text{if } 0 < x \le \frac{1}{4} \\ \frac{1}{3} & \text{if } \frac{1}{4} < x < \frac{1}{2} \end{cases}$$

$$S(x) = T(x) = \frac{1}{2} - x \quad \text{if } 0 < x < \frac{1}{2}$$

Then $A(X) = B(X) = \left\{\frac{1}{4}, \frac{1}{3}\right\}$ $S(X) = T(X) = \left(0, \frac{1}{2}\right)$ while so that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ proving the condition (ii) of Theorem (A). Clearly (X,d) is not a complete metric space. It is easy to prove that the iterated sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}\ldots$, converges to $\frac{1}{4}$ if $0 < x \le \frac{1}{4}$.

III. RESULTS AND DISCUSSION

Theorem (B): Suppose A, B, S and T are four self-maps of metric space (X,d) such that

(i) $A(X)\subseteq T(X)$, $B(X)\subseteq S(X)$

(ii) One of the A,B,S,T is continuous

(iii) The pairs (A,S) and (B,T) are compatible of type (P)

(iv) $d(Ax,By)^2$

$$\leq c_1 \max\{d(Sx,Ax)^2, d(Ty,By)^2, d(Sx,Ty)^2\}$$
$$+c_2 \max\{d(Sx,Ax)d(Sx,By), d(Ax,Ty)d(By,Ty)\}$$
$$+c_3 \{d(Sx,By)d(Ty,Ax)\}$$

where $c_1,c_2,c_3, \geq 0$, $c_1+2c_2<1$ and $c_1+c_3< 1$

Further if

(v) The sequence Ax_0, Bx_1, Ax_2, Bx_3….. Ax_{2n}, Bx_{2n+1}…. converges to $z\in X$ then A, B,S and T has a unique common fixed point $z\in X$.

Proof : From condition v of Theorem (B), we have Ax_{2n}, Bx_{2n+1} converges to z as $n\to\infty$.

Suppose that S is continuous, then $SSx_{2n}\to Sz$ and $SAx_{2n}\to Sz$ as $n\to\infty$.

Since the pair (A,S) is compatible type (p), we get

$$\lim_{n\to\infty} d(AAx_{2n}, SSx_{2n}) = 0.$$

This gives $AAx_{2n} \to Sz$ as $n\to\infty$.

Put $x=Ax_{2n}$, $y=x_{2n+1}$ in (iv) of theorem(B), we get

$$[d(Sz,z)]^2$$
$$= [d(AAx_{2n},Bx_{2n+1})]^2$$
$$\leq c_1 \max\{[d(SAx_{2n},AAx_{2n})]^2,[d(Bx_{2n+1},Tx_{2n+1})]^2, [d(Bx_{2n+1},Tx_{2n+1})]^2\}$$
$$+c_2 \max\{d(SAx_{2n},AAx_{2n})d(SAx_{2n},Bx_{2n+1}), d(AAx_{2n},Tx_{2n+1}(Bx_{2n+1},Tx_{2n+1})\}$$
$$+c_3\{d(SAx_{2n},Bx_{2n+1})d(Tx_{2n+1},AAx_{2n})\}$$

Letting $n \to \infty$ we get

$$[d(Sz,z)]^2=[d(Sz,z)]^2$$
$$\leq c_1 \max\{[d(Sz,Sz)]^2,[d(z,z)]^2,[d(Sz,z)]^2$$
$$+c_2 \max\{d(Sz,Sz)d(Sz,z),d(Sz,z)d(z,z)\}$$
$$+c_3\{d(Sz,z)d(z,Sz)\}$$

$$[d(Sz,z)]^2\leq(c_1+c_2)\,[d(Sz,z)]^2$$
$$[d(Sz,z)]^2[1-(c_1+c_2)]\leq 0$$

Since $c_1+c_2<1$ we get $[d(Sz,z)]^2$ or Sz=z.

Put x=z and $y=x_{2n+1}$ in (iv) of theorem (B), then

$$[d(Az,z)]^2=[d(Az,Bx_{2n+1})]^2$$
$$\leq c_1 \max\{[d(Sz,Az)]^2,[d(Bx_{2n+1},Tx_{2n+1})]^2, [d(Sz,Tx_{2n+1})]^2\}$$
$$+c_2 \max\{d(Sz,Az)d(Sz,Bx_{2n+1}), d(Az,Tx_{2n+1})d(Bx_{2n+1},Tx_{2n+1})\}$$
$$+c_3\{d(Sz,Bx_{2n+1})d(Tx_{2n+1},Az)\}$$

Letting $n \to \infty$ we get

$$[d(Az,z)]^2=$$
$$\leq c_1 \max\{[d(z,Az)]^2,[d(z,z)]^2,[d(z,z)]^2$$
$$+c_2 \max\{d(z,Az)d(z,z),d(Az,z)d(z,z)\}$$
$$+c_3\{d(z,z)d(z,Az)\}$$

$$[d(Az,z)]^2(1-c_2)\leq 0$$

Since $c_1+c_3<1$ and $c_1+2c_2<1$ we get

$[d(Az,z)]^2=0$ or Az=z

Then we get z=Az=Sz.

Since $A(X)\subseteq T(X)$ we get there exists $u\in X$ such that z=Az=Tu.

We prove Bu=z

$$[d(z,Bu)]^2=[d(Az,Bu)]^2$$
$$\leq c_1 \max\{[d(Sz,Az)]^2,[d(Bu,Tu)]^2,[d(Sz,Tu)]^2$$
$$+c_2 \max\{d(Sz,Az)d(Sz,Bu),d(Az,Tu)d(Bu,Tu)\}$$
$$+c_3\{d(Sz,Bu)d(Tu,Az)\}$$

Letting $n \to \infty$ we get

$$[d(z,Bu)]^2 \leq c_1 \max\{[d(z,z)]^2,[d(Bu,z)]^2,[d(z,z)]^2$$
$$+c_2 \max\{d(z,z)d(z,Bu),d(z,z)d(Bu,z)\}$$
$$+c_3\{d(z,Bu)d(z,z)\}$$

$$[d(z,Bu)]^2[1-c_1]\leq 0.$$

Since $c_1+c_3<1$ and $c_1+2c_2<1$ we get

$[d(z,Bu)]^2=0$ or Bu=z.

Therefore z=Bu=Tu.

Since the pair (B,T) is compatible type(P) and z= Bu=Tu, we get d(BBu,TTu)=0 or Bz=Tz

Put x=z and y= z in (iv) of theorem(B), then

$$[d(z,Bz)]^2=[d(Az,Bz)]^2$$

$$\leq c_1 \max\{[d(Sz,Az)]^2,[d(Bz,Tz)]^2,[d(Sz,Tz)]^2$$

$$+c_2 \max\{d(Sz,Az)d(Sz,Bz),d(Az,Tz)d(Bz,Tz)\}$$

$$+c_3\{d(Sz,Bz)d(Tz,Az)\}$$

$$[d(z,Bz)]^2 \leq c_1$$

$$\max\{[d(z,z)]^2,[d(Bz,Bz)]^2,[d(z,Bz)]^2$$

$$+c_2 \max\{d(z,z)d(z,Bz),d(z,Bz)d(Bz,Bz)\}$$

$$+c_3\{d(z,Bz)d(Bz,z)\}$$

$[d(z,Bz)]^2[1-(c_1+c_3)]\leq 0$ since $c_1+c_3<1$ and $c_1+2c_2<1$ we get $[d(z,Bz)]^2=0$ or $Bz=z$.

Therefore z=Bz=Tz. Since z=Az=Bz=Sz=Tz. So z is a common fixed point of A, B, S and T. The uniqueness of common fixed point can be easily proved.

Remark : From the example given above, Clearly the pairs (A,S) and (B,T) are not compatible, but they are compatible of type (P).Also the inequality (iv) of theorem(B) holds for the values of $c_1+2c_2 < 1$ and $c_1+c_3 < 1$. We note that X is not a complete metric space and it is easy to prove that the Iterated sequence Ax_0, Bx_1 Ax_2, Bx_3,.., Ax_{2n}, Bx_{2n+1},……. converges to the point 1/4 which is a common fixed point of A, B, S and T. In fact, 1/4 is the unique common fixed point of A, B, S and T.

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