© 2014 IJSRSET | Volume 1 | Issue 1 | Print ISSN: 2395-1990 | Online ISSN : 2394-4099
Themed Section: Engineering and Technology

Associated Left k-Fibonacci Numbers, Generating Function and their Identity

Mansukh P. Arvadia

Shri U P Arts, Smt. M G Panchal Science & Shri V L Shah Commerce College, Pilvai, Gujarat, India

ABSTRACT

In this paper we define the associated left $k-$Fibonacci numbers, generating function and we give interesting identities of it.

Keywords: Associated $k-$Fibonacci numbers, associated left $k-$Fibonacci numbers, generating function, recurrence relation.

I. INTRODUCTION

The Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$, starting with the integer pair 0 and 1 and each Fibonacci number is obtained by the sum of the two preceding it. We define as $F_n = F_{n-1} + F_{n-2};\ n \geq 2$ [5,8]. The first few terms of the Fibonacci numbers are 0,1,1,2,3,5,8,13,21,34, 55, 89, ….. The Fibonacci sequence has been generalized in many ways, some by preserving the initial conditions, and others by preserving the recurrence relation [5,6,7,9].

In [1] Alvaro H. Salas defined the sequence $\{A_{k,n}\}_{n=0}^{\infty}$ associated to $\{F_{k,n}\}_{n=0}^{\infty}$ as $A_{k,n} = F_{k,n} + F_{k,n-1}$ where $A_{k,0} = 1$ for $n=1,2,3,\ldots$. He then defines associated k- Fibonacci numbers by recurrence relation

$$A_{k,n} = kA_{k,n-1} + A_{k,n-2};\ n \geq 2 \text{ where } A_{k,0} = 1.$$

In this paper we define *associated left k- Fibonacci sequence* $\{A_{k,n}^L\}$ and prove some interesting properties of *associated left k- Fibonacci numbers* [2,3]. Despite its simple appearance, this sequence contains a wealth of subtle and fascinating properties. In this paper we explore several of the fundamental identities related with $A_{k,n}^L$.

II. The associated left k-Fibonacci numbers:

Definition: The sequence of *associated left k-Fibonacci numbers* $\{A_{k,n}^L\}$ associate to left k-Fibonacci sequence $\{F_{k,n}^L\}$ [4] is defined as

$$A_{k,0}^L = 1 \text{ and}$$

$$A_{k,n}^L = F_{k,n}^L + F_{k,n-1}^L,\ n = 1,2,3\ldots.$$

We observe that the expression $A_{k,n}^L$ is the sum of the two consecutive *left k-Fibonacci numbers* $F_{k,n}^L$ and its predecessor $F_{k,n-1}^L$. The members of the sequence $\{A_{k,n}^L\}$ will be called *associated left k-Fibonacci numbers*. An equivalent definition for the sequence $\{A_{k,n}^L\}$ is

$$A_{k,n}^L = \begin{cases} 1 & ,if\ \ n=0 \\ 1 & ,if\ \ n=1 \\ (k+1)F^L{}_{k,n-1} + F^L{}_{k,n-2} & ,if\ \ n \geq 2 \end{cases}$$

Observe that

$$A_{k,n}^L = F_{k,n}^L + F_{k,n-1}^L = kF_{k,n-1}^L + F_{k,n-2}^L + kF_{k,n-2}^L + F_{k,n-3}^L$$

$$= k(F^L_{k,n-1} + F^L_{k,n-2}) + F^L_{k,n-2} + F^L_{k,n-3}$$
$$= kA^L_{k,n-1} + A^L_{k,n-2}$$

This allows defining recursively the sequence of associated left k-Fibonacci numbers as follows:

$$A^L_{k,n} = kA^L_{k,n-1} + A^L_{k,n-2}$$

III. Some Interesting identities of $\{A^L_{k,n}\}$.

One of the purposes of this paper is to develop many identities and results. We use the technique of induction as a useful tool in proving many of these identities and theorems involving Fibonacci numbers.

We derived the interesting extended reduction formula for $A^L_{k,n}$.

Identity 3.1 (Reduction Formula)

$$A^L_{k,m+n} = A^L_{k,m-1}A^L_{k,n} + A^L_{k,m}A^L_{k,n+1} - A^L_{k,m+n-1}.$$

Proof Let m be a fixed integer and we proceed by inducting on n.

For $n = 1$, we have

$$A^L_{k,m+1} = A^L_{k,m-1}A^L_{k,1} + A^L_{k,m}A^L_{k,2} - A^L_{k,m}$$

$$A^L_{k,m+1} = A^L_{k,m-1}(1) + A^L_{k,m}(k+1) - A^L_{k,m}$$

$$A^L_{k,m+1} = kA^L_{k,m} + A^L_{k,m-1}, \quad (\because A^L_{k,1} = 1,\ A^L_{k,2} = k+1)$$

This is obvious.

Now let us assume that the result is true for $n = 1, 2, 3, \cdots, t$;

$$A^L_{k,m+t} = A^L_{k,m-1}A^L_{k,t} + A^L_{k,m}A^L_{k,t+1} - A^L_{k,m+t-1} \text{ and}$$

$$A^L_{k,m+(t-1)} = A^L_{k,m-1}A^L_{k,t-1} + A^L_{k,m}A^L_{k,t} - A^L_{k,m+t-2}$$

We will show that it holds for $n = t+1$, also from above two equation, we have

$$A^L_{k,m+t+1} = kA^L_{k,m+t} + A^L_{k,m+(t-1)}$$

$$= k(A^L_{k,m-1}A^L_{k,t} + A^L_{k,m}A^L_{k,t+1}) + (A^L_{k,m-1}A^L_{k,t-1} + A^L_{k,m}A^L_{k,t})$$

$$= A^L_{k,m-1}(kA^L_{k,t} + A^L_{k,t-1}) + A^L_{k,m}(kA^L_{k,t+1} + A^L_{k,t}) - (kA^L_{k,m+t-1} + A^L_{k,m+t}$$

$$= A^L_{k,m-1}A^L_{k,t+1} + A^L_{k,m}A^L_{k,(t+1)+1} - A^L_{k,m+t} = A^L_{k,m+(t+1)}$$

Thus the result is true for all $n \in N$. This proves the result.

It is often useful to extend the sequence of *associated left k- Fibonacci numbers* backward with negative subscripts. In fact if we try to extend the *associated left k- Fibonacci sequence* backward still keeping to the same rule, we get the following:

n	$A^L_{k,n}$
-1	$1-k$
-2	$1-k+k^2$
-3	$1-2k+k^2-k^3$
-4	$1-2k+3k^2-k^3+k^4$
-5	$1-3k+3k^2-4k^3+k^4-k^5$

Thus the sequence of *associated left k- Fibonacci numbers* is bilateral sequence, since it can be extended infinitely in both directions.

We next prove the divisibility property for $A^L_{k,n}$.

Identity 3.2 $A^L_{k,m} \mid A^L_{k,mn}$ for all non-zero integers m, n .

Proof: Let m be fixed and we will proceed by inducting on n.

For $n = 1$. Then it is clear that $A^L_{k,m} \mid A^L_{k,m}$.

$\therefore$ The result is true for $n = 1$. Assume that the result is true for all $n = 1, 2, 3, \cdots, t$.

Thus $A^L_{k,m} \mid A^L_{k,mt}$ holds by assumption.

To prove the result is true for $n = t+1$. Using lemma 4.3.6, we get

$$A^L_{k,m(t+1)} = A^L_{k,mt+m}$$
$$= A^L_{k,mt-1}A^L_{k,m} + A^L_{k,mt}A^L_{k,m+1} - A^L_{k,mt+m-1}$$

$$= A^L_{k,mt-1}A^L_{k,m} + A^L_{k,mt}A^L_{k,m+1} - (A^L_{k,mt-1}A^L_{k,m-1} + A^L_{k,mt}A^L_{k,m} - A^L_{k,mt+m-2})$$

Continuously expand this expression; as by assumption $A^L_{k,m} \mid A^L_{k,mt}$

$A^L_{k,m}$ divides the entire right side of the equation.

Hence $A^L_{k,m} \mid A^L_{k,m(t+1)}$. Thus result is true for all $n \geq 1$.

We next derive the formula for the sum of the squares of first n *associated left k- Fibonacci numbers*.

Identity 3.3

$$A^{L\ 2}_{k,1} + A^{L\ 2}_{k,2} + A^{L\ 2}_{k,3} + \cdots + A^{L\ 2}_{k,n} = \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - 1).$$

Proof: Since we have $A^L_{k,m} = \frac{1}{k}\left(A^L_{k,m+1} - A^L_{k,m-1}\right)$

We observe that $A^{L\ 2}_{k,m} = A^L_{k,m}A^L_{k,m}$

$$= A^L_{k,m}\left[\frac{1}{k}\left(A^L_{k,m+1} - A^L_{k,m-1}\right)\right] = \frac{1}{k}\left(A^L_{k,m}A^L_{k,m+1} - A^L_{k,m}A^L_{k,m-1}\right)$$

Replacing $m = 1, 2, 3 \cdots n$, we have

$$A^{L\ 2}_{k,1} = \frac{1}{k}(A^L_{k,1}\,A^L_{k,2} - A^L_{k,1}\,A^L_{k,0})$$

$$A^{L\ 2}_{k,2} = \frac{1}{k}(A^L_{k,2}\,A^L_{k,3} - A^L_{k,1}A^L_{k,2})$$

$$A^{L\ 2}_{k,3} = \frac{1}{k}(A^L_{k,3}\,A^L_{k,4} - A^L_{k,2}A^L_{k,3})$$

$$A^{L\ \ 2}_{k,n-1} = \frac{1}{k}(A^L_{k,n-1}\,A^L_{k,n} - A^L_{k,n-2}A^L_{k,n-1})$$

$$A^{L\ 2}_{k,n} = \frac{1}{k}(A^L_{k,n}\,A^L_{k,n+1} - A^L_{k,n-1}A^L_{k,n})$$

Adding all these equations, we get

$$A^{L\ 2}_{k,1} + A^{L\ 2}_{k,2} + A^{L\ 2}_{k,3} + \cdots + A^{L\ 2}_{k,n} = \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - A^L_{k,1}\,A^L_{k,0})$$

$$= \frac{1}{k}(A^L_{k,n}A^L_{k,n+1} - 1).$$

The following result follows immediately from this lemma.

Corollary 3.4 $A^L_{k,n}A^L_{k,n+1} \equiv 1\left(mod\,k\right)$.

Proof: We use Mathematical Induction to prove the result.

For $n = 1$, we have

$$A^L_{k,1}A^L_{k,2} = 1\left(1+k\right) = 1 + k \equiv 1\left(mod\,k\right)$$

Suppose it is true for $n = r$, Thus $F^R_{k,r}F^R_{k,r+1} \equiv 1\left(mod\,k\right)$ holds.

Now, $A^L_{k,r+1}A^L_{k,r+2} = A^L_{k,r+1}\left(kA^L_{k,r+1} + A^L_{k,r}\right)$.

$$= kA^{L\ \ 2}_{k,r+1} + A^L_{k,r+1}A^L_{k,r} \equiv 1\left(mod\,k\right)$$

So the result is true for $n = r+1$ also. This proves the result for all integers *n*.

Similarly, we can prove the second result.

We finally prove the extended Cassini's identity.

Identity 3.5 (Cassini's identity**)**

$$A^L_{k,n+1}A^L_{k,n-1} - A^{L\ 2}_{k,n} = k(-1)^{n+1}.$$

Proof: We have

$$A^L_{k,n+1}A^L_{k,n-1} - A^{L\ 2}_{k,n} = (kA^L_{k,n} + A^L_{k,n-1})A^L_{k,n-1} - A^{L\ 2}_{k,n}$$

$$= kA^L_{k,n}A^L_{k,n-1} - A^{L\ 2}_{k,n} + A^{L\ \ 2}_{k,n-1}$$

$$= A^L_{k,n}(kA^L_{k,n-1} - A^L_{k,n}) + A^{L\ \ 2}_{k,n-1}$$

$$= -A^L_{k,n}A^L_{k,n-2} + A^{L\ \ 2}_{k,n-1}$$

$$= (-1)(A^L_{k,n}A^L_{k,n-2} - A^{L\ \ 2}_{k,n-1}).$$

Repeating the same process successively for right side, we get

$$\therefore A^L_{k,n+1}A^L_{k,n-1} - A^{L\ 2}_{k,n} = (-1)^1(A^L_{k,n}A^L_{k,n-2} - A^{L\ \ 2}_{k,n-1})$$

$$= (-1)^2(A^L_{k,n-1}A^L_{k,n-3} - A^{L\ \ 2}_{k,n-2})$$

$$= (-1)^3(A^L_{k,n-2}A^L_{k,n-4} - A^{L\ \ 2}_{k,n-3})$$

$$= (-1)^n(A^L_{k,1}A^L_{k,-1} - A^{L\ 2}_{k,0})$$

$$= (-1)^n(1(1-k) - 1)$$

$$= k(-1)^{n+1}.$$

IV. Generating function of associated left k-Fibonacci numbers

Generating function provide a powerful tool for solving linear homogeneous recurrence relation with constant coefficients. In 1718, the French mathematician Abraham De Moivre (1667−1754) invented generating functions in order to solve the Fibonacci recurrence relation. Let $a_0, a_1, a_2, \cdots$ be a sequence of real numbers. Then the function

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n + \cdots$$

is called the generating function for the sequence $\{a_n\}$.

We can also define generating functions for the finite sequence $a_0, a_1, \ldots, a_n$ by letting $a_i = 0$. For $i > n$. Thus $g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is the generating function for the finite sequence $a_0, a_1, \ldots, a_n$. For example, function $g(x) = 1 + 2x + 3x^2 + \cdots + (n+1)x^n + \cdots$ is the generating function for the sequence of positive integer, where as $f(x) = 1 + 3x + 6x^2 + \cdots + \frac{n(n+1)}{2}x^n$ is the generating function for the sequence of triangular numbers $1, 3, 6, 10, \ldots$

two generating function $f(x) = \sum_0^\infty a_n x^n$ and $\sum_0^\infty b_n x^n$ be two generating functions. Then $f(x) + g(x) = \sum_0^\infty (a_n + b_n)x^n$ and $f(x)g(x) = \sum_{n=0}^\infty (\sum_{i=0}^\infty a_i b_{n-i})x^n$

The *associated left k-Fibonacci number* which is defined as $A_{k,n+1}^L = kA_{k,n}^L + A_{k,n-1}^L,\ n \geq 1$ with initial condition $A_{k,0}^L = 1$ is a second order difference equation with constant coefficient. Therefore, it has the characteristic equation $x^2 - kx - 1 = 0$.

Theorem 4.1 The generating function for the generalized *associated left k- Fibonacci sequence* $\left\{A_{k,n}^L\right\}_{n=0}^\infty$ is given by $f(x) = \dfrac{1 + x - kx}{1 - kx - x^2}$.

Proof: We begin with the formal power series representation of generating function for $\left\{A_{k,n}^L\right\}$ that is for $\left\{g_n\right\}$.

$$f(x) = \sum_{m=0}^\infty A_{k,m}^L x^m = \sum_{m=0}^\infty g_m x^m = g_0 + g_1 x + g_2 x^2 + \cdots$$

$$= 1 + (1)x + (kg_1 + g_0)x^2 + (kg_2 + g_1)x^3 + (kg_3 + g_2)x^4 + \cdots$$

$$= 1 + x + kx(g_1 x + g_2 x^2 + \cdots) + x^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= 1 + x + kx(g_0 + g_1 x + g_2 x^2 + \cdots) - kx + x^2(g_0 + g_1 x + g_2 x^2 + \cdots)$$

$$= 1 + x + kxf(x) + x^2 f(x) - kx$$

$$\therefore\ (1 - kx - x^2) f(x) = 1 + x - kx \Rightarrow f(x) = \frac{1 + x - kx}{1 - kx - x^2}$$

.

This is the generating function for the generalized *associated left k- Fibonacci sequence* $\left\{A_{k,n}^L\right\}_{n=0}^\infty$.

V. CONCLUSION

A new generalized associated left k-Fibonacci sequence has been introduced and deducted their identities, generating function and results.

VI. ACKNOWLEDGEMENT

This work has been supported in part by UGC Minor Project number F. 47-901/14 (WRO) from Ministry of Human resource Development, Govt. of India.

VII. REFERENCES

[1]. Alvaro H. Salas: About k-Fibonacci Numbers and their Associated Numbers, International Mathematical Forum, Vol. 6 (2011), Issue 50, page 2473-2479.

[2]. Arvadia Mansukh P., Shah Devbhadra V.: Right k-Fibonacci sequence and related identities, International Research Journal of Mathematics, Engineering & IT, Volume-2, Issue-4 (2015), 25-39.

[3]. Arvadia Mansukh P., Shah Devbhadra V.: Left k-Fibonacci Sequence and Related Identities, Journal Club for Applied Sciences, V-2, I-1(2015), 20-26.

[4]. Cennet Bolat, Hasan Kose: On the properties of k-Fibonacci Numbers, Int. J. Contemp. Math. Sciences, Vol. 5, (2010), No. 22, 1097-1105.

[5]. Falcon S: A simple proof of an interesting Fibonacci generalization, International Journal of Mathematics Education, Science and Technology, 35, No. 2, (2004), 259-261.

[6]. Falcon S., Plaza, A :On the Fibonacci k-numbers, Chaos, Solitons & Fractals, 32 (5) (2007) 1615-24.

[7]. Falcon S., Plaza, A :The k-Fibonacci sequence and Pascal 2-triangle, Chaos, Solitons & Fractals (33) (1) (2007) 38-49.

[8]. Koshy Thomas: Fibonacci and Lucas Numbers with applications, John Wiley and Sons, Inc., N. York, 2001.

[9]. Marcia Edson, Omer Yayenie: A new generalization of Fibonacci sequence and extended Binet's formula, Integers, Volume 9, Issue 6, (2009), 639-654.