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Ulam-Hyers Stability of Additive and Reciprocal Functional Equations: Direct and Fixed Point Methods

M. Arunkumar[1], A. Vijayakumar[2], S. Karthikeyan[3]
[1] Department of Mathematics, Government Arts College, Tiruvannamalai, Tamilnadu, India
[2,3]Department of Mathematics, R.M.K. Engineering College, Kavarapettai, Tamilnadu, India

ABSTRACT

In this paper, the authors established the generalized Ulam - Hyers stability of additive functional equation

$$f(x)=\sum_{l=1}^{n}\left(\frac{f\left(x+ly_l\right)+f\left(x-ly_l\right)}{2n}\right)$$

which is originating from arithmetic mean of n consecutive terms of an arithmetic progression in Intuitionistic fuzzy normed spaces and reciprocal functional equation

$$h\left(\frac{2x}{n}\right)=\sum_{l=1}^{n}\left(\frac{h(x+ly_l)h(x-ly_l)}{h(x+ly_l)+h(x-ly_l)}\right)$$

originating from n-consecutive terms of a harmonic progression in Non - Archimedean Fuzzy $\varphi-2-$ normed spaces using direct and fixed point methods. Applications of the above functional equations are also given.

Keywords: Additive functional equation, Reciprocal functional equation, generalized Ulam-Hyers stability, Intuitionistic fuzzy normed spaces, Non - Archimedean Fuzzy $\varphi-2-$ normed spaces, fixed point method.

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1. INTRODUCTION

In 1940, S.M. Ulam [47] introduced the stability of functional equations. Next year 1941, D. H. Hyers [16] gave first confirmatory answer to the Ulam question for Banach spaces. In 1978, Hyers theorem was generalized by Th.M. Rassias [37]. Gajda [12] answered the question for the case $p>1$ in the year 1991, which was raised by Rassias. This stability results is known as generalized Hyers-Ulam stability of functional equations (see [1, 2, 14, 20, 22, 26, 38]). During the years 1982–1994, Rassias [32-36] investigated the Ulam stability problem for different mappings involving a product of different powers of norms. Recently, Rassias gave the mixed product sum of powers of norms control function [39]. We also refer the readers to the books: P. Czerwik [7] and D.H. Hyers, G. Isac and Th.M. Rassias [17].

In 2003, V. Radu [31] introduced a new method, successively developed in [8-10], to obtaining the existence of the exact solutions and the error estimations, based on the fixed point alternative. The stability of several functional equations have been extensively investigated by a number of mathematicians and there are many interesting results concerning this problem (see [3, 4, 21, 23-25, 40, 41]).

In this paper, the authors proved the generalized Ulam - Hyers stability of an additive functional equation

$$f(x)=\sum_{l=1}^{n}\left(\frac{f\left(x+ly_l\right)+f\left(x-ly_l\right)}{2n}\right) \tag{1.1}$$

which is originating from arithmetic mean of n consecutive terms of an arithmetic progression in Intuitionistic fuzzy normed spaces, and reciprocal functional equation

$$h\left(\frac{2x}{n}\right)=\sum_{l=1}^{n}\left(\frac{h(x+ly_l)h(x-ly_l)}{h(x+ly_l)+h(x-ly_l)}\right) \tag{1.2}$$

Which is originating from *n*-consecutive terms of an harmonic progression in Non - Archimedean Fuzzy $\varphi-2-$ normed spaces using direct and fixed point methods. Applications of the above functional equations are also investigated.

2. PRELIMINARIES OF INTUITIONISTIC FUZZY NORMED AND NON-ARCHIMEDEAN FUZZY $\varphi-2-$ NORMED SPACES

In this section, we give some basic definitions and lemmas for the main results in this article.

Definition 2.1. Let μ and ν be membership and nonmembership degree of an intuitionistic fuzzy set from $X\times(0,+\infty)$ to [0,1] such that $\mu_x(t)+\nu_x(t)\le 1$ for all $x\in X$ and all $t>0$. The triple $(X,P_{\mu,\nu},M)$ is said to be an *intuitionistic fuzzy normed space* (briefly IFN-space) if X is a vector space, M is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X\times(0,+\infty)\to L^*$ satisfying the following conditions: for all $x,y\in X$ and $t,s>0$,

$(IFN1)$ $P_{\mu,\nu}(x,0) = 0_{L^*}$; $(IFN2)$ $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x=0$;

$(IFN3)$ $P_{\mu,\nu}(\alpha x,t) = P_{\mu,\nu}\left(x,\frac{t}{|\alpha|}\right)$ for all $\alpha\ne 0$; $(IFN4)$ $P_{\mu,\nu}(x+y,t+s) \ge_{L^*} \mathrm{M}\left(P_{\mu,\nu}(x,t),P_{\mu,\nu}(y,s)\right)$.

In this case, $P_{\mu,\nu}$ is called an *intuitionistic fuzzy norm*. Here $P_{\mu,\nu}(x,t)=(\mu_x(t),\nu_x(t))$.

Example 2.2. Let $(X,\|.\|)$ be a normed space. Let $T(a,b)=(a,b\ \min\ (a_2+b_2,1))$ for all $a=(a_1,a_2),b=(b_1,b_2)\in L^*$ and μ,ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,\nu}(x,t)=(\mu_x(t),\nu_x(t))=\left(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\right),\forall t\in R^+.$$

Then $(X,P_{\mu,\nu},T)$ is an IFN-space.

Definition 2.3. A sequence $\{x_n\}$ in an IFN-space $(X,P_{\mu,\nu},T)$ is called a *Cauchy sequence* if, for any $\varepsilon>0$ and $t>0$, there exists $n_0\in N$ such that $P_{\mu,\nu}(x_n-x_m,t)>_{L^*}(N_s(\varepsilon),\varepsilon)$, $\forall n,m\ge n_0$, where N_s is the standard negator.

Definition 2.4. The sequence $\{x_n\}$ is said to be *convergent* to a point $x\in X$

(denoted by $x_n\xrightarrow{P_{\mu,\nu}}x$) if $P_{\mu,\nu}(x_n-x,t)\to 1_{L^*}$ as $n\to\infty$ for every $t>0$.

Definition 2.5. An IFN-space $(X,P_{\mu,\nu},T)$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x\in X$.

For further details about IFN space one can see ([5, 6, 1, 17-19, 30, 43, 44- 46, 48-50]).

Based on [15], some basic definitions and notations in $\varphi-2-$ normed spaces is provided.

Definition 2.7 A $t-$norm $\Diamond$ is a two place function $\Diamond:[0,1]\bullet\times[0,1]\to[0,1]$ which is associative, commutative, non decreasing in each place and such that $a\Diamond 1=a$, for all $a\in[0,1]$.

Definition 2.8 Let φ be a function defined on the real field P into itself with the following properties :

- $\varphi(-t)=\varphi(t),$ for every $t\in\mathbb{R}$;
- $\varphi(1)=1;$
- φ is strict increasing and continuous on $(0,\infty)$;
- $\lim_{\alpha\to 0}\varphi(\alpha)=0$ and $\lim_{\alpha\to\infty}\varphi(\alpha)=\infty$.

Example 2.9 The functions

- $\varphi(\alpha)=|\alpha|$ for every $\alpha\in\mathbb{R}$;
- $\varphi(\alpha^p)=|\alpha|^p$ for every $p\in\mathbb{R}_+$.

Definition 2.10 [7] Let L be a linear space over the field R of a dimension greater than one and let N be a mapping defined on $L\times L\times[0,\infty)$ with values into $[0,1]$ satisfying the following conditions: for all $x,y,z\in L$ and $s,t\in[0,\infty)$

(NAF1) $N(x,y,0)=0$; (NAF2) $N(x,y,t)=1$, for all $t>0$ if and only if x,y are linear dependent;

(NAF3) $N(x,y,t)=N(y,x,t)$ for all $x,y\in L$, and $t>0$; (NAF4) $N(x+y,z,\max(t,s))\geq \min\left(N(x,z,t)\Diamond N(y,z,s)\right)$;

(NAF5) $N(x,y,\cdot):[0,\infty)\to[0,1]$ is left continuous. (NAF6) $N\left(\alpha x,y,t\right)=N\left(x,y,\frac{t}{\varphi(\alpha)}\right),\alpha\in R.$

The triple $(L,N,\Diamond)$ will be called a non-Archimedean fuzzy $\varphi-2-$normed space.

Example 2.11 Let $\left(L,\|\cdot,\cdot\|\right)$ be a non-Archimedean fuzzy $\varphi-2-$normed space. Then

$$N\left(x,t\right)=\begin{cases}\frac{t}{t+\|x\|}, & t>0,\ x\in X,\\ 0, & t\leq 0,\ x\in X\end{cases}$$

Then $(L,N,\Diamond)$ is a non-Archimedean fuzzy $\varphi-2-$normed space.

Definition 2.12 Let $(L,N,\Diamond)$ be a non-Archimedean fuzzy $\varphi-2-$normed space. Let x_n be a sequence in L. Then $\{x_n\}$ is said to be convergent if there exists $x\in X$ such that

$$\lim_{n\to\infty}N(x_n-x,a,t)=1$$

for all $a\in L$ and $t>0$. In that case, x is called the limit of the sequence x_n and we denote it by

$$N-\lim_{n\to\infty}x_n=x.$$

Definition 2.13 A sequence x_n in L is called Cauchy if $N(x_{n+p}-x_n,a,t)=1$ for all $a\in L$, $p>0$ and $t>0$.

Definition 2.14 Every convergent sequence in a non-Archimedean fuzzy $\varphi-2-$normed space is a Cauchy sequence. If every Cauchy sequence is convergent, then the non-Archimedean fuzzy $\varphi-2-$ normed space is called a non-Archimedean fuzzy $\varphi-2-$ Banach space.

For further details about non-Archimedean fuzzy $\varphi-2-$normed space one can see ([11, 15, 16, 27, 29, 42])

Definition 2.15 Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a generalized metric on X if d satisfies the following:

(1) $d\left(x,y\right)=0$ if and only if $x=y$; (2) $d\left(x,y\right)=d\left(y,x\right)$ for all $x,y\in X$;

(3) $d\left(x,y\right)\leq d\left(x,z\right)+d\left(z,y\right)$ for all $x,y,z\in X$.

For explicit later use, we recall a fundamental result in fixed point theory.

Theorem 2.16 *[28](The alternative of fixed point)* Suppose that for a complete generalized metric space (X,d) and a strictly contractive mapping $T:X\to X$ with Lipschitz constant L. Then, for each given element $x\in X$, either

$(B1)$ $d(T^nx,T^{n+1}x)=\infty\ \ \forall\ n\geq 0,$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^nx,T^{n+1}x)<\infty$ for all $n\geq n_0$;

(ii) The sequence (T^nx) is convergent to a fixed point y^* of T;

(iii) y^* is the unique fixed point of T in the set $Y=\{y\in X: d(T^{n_0}x,y)<\infty\}$;

(iv) $d(y^*,y)\leq\frac{1}{1-L}\ d(y,Ty)$ for all $y\in Y$.

Throughout this paper we define a mapping $Df:X\to Y$ by

$$Df\left(x, y_1, y_2, ..., y_n\right) = f(x) - \sum_{l=1}^{n}\left(\frac{f\left(x+ly_l\right)+f\left(x-ly_l\right)}{2n}\right)$$

for all $x, y_1, y_2, ..., y_n \in X$ and a mapping $DH : X \to Y$ such that

$$DH(x, y_1, y_2, \cdots, y_n) = h\left(\frac{2x}{n}\right) - \sum_{l=1}^{n}\left(\frac{h(x+ly_l)h(x-ly_l)}{h(x+ly_l)+h(x-ly_l)}\right)$$

for all $x, y_1, y_2, \cdots, y_n \in X$.

3. INTUITIONISTIC FUZZY NORMED SPACE STABILITY: DIRECT METHOD

In this section, the authors present the generalized Ulam - Hyers stability of the functional equation (1.1) in IFN - space using direct method.

Throughout this section, let us consider X, $(Z, P_{\mu,\nu}, M)$ and $(Y, P'_{\mu,\nu}, M)$ are linear space, Intuitionistic fuzzy normed space and Complete Intuitionistic fuzzy normed space respectively.

Theorem 3.1. Let $\delta \in \{-1,1\}$ be fixed and let $\xi : X^{n+1} \to Z$ be a mapping such that for some b with $0 < \left(\frac{b}{2}\right)^{\gamma} < 1$

$$P'_{\mu,\nu}\left(\xi\left(2^{\delta}x, 2^{\delta}x, 0, \cdots, 0\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(b^{\delta}\xi\left(x, x, 0, \cdots, 0\right), r\right), \tag{3.1}$$

for all $x \in X$ and all $r > 0,\ b > 0$ and

$$\lim_{k\to\infty} P'_{\mu,\nu}\left(\xi\left(2^{\delta k}x, 2^{\delta k}y_1, \cdots, 2^{\delta k}y_n\right), 2^{\delta k}r\right) = 1_{L^*} \tag{3.2}$$

for all $x, y_1, y_2, ..., y_n \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(Df\left(x, y_1, y_2, ..., y_n\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, y_1, y_2, ..., y_n\right), r\right) \tag{3.3}$$

for all $x, y_1, y_2, ..., y_n \in X$ and all $r > 0$. Then the limit

$$P_{\mu,\nu}\left(A(x) - \frac{f(2^{\delta k}x)}{2^{\delta k}}, r\right) \to 1_{L^*} \quad as \quad k \to \infty, r > 0 \tag{3.4}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping satisfying (1.1) and

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), rn|2-b|\right) \tag{3.5}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\delta = 1$. Replacing $\left(x, y_1, y_2, ..., y_n\right)$ by $\left(x, x, 0, ..., 0\right)$ in (3.3), we arrive

$$P_{\mu,\nu}\left(2nf(x) - nf(2x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), r\right)$$

for all $x \in X$ and all $r > 0$. Using $(IFN3)$ in the above equation, we get

$$P_{\mu,\nu}\left(f(x) - \frac{f(2x)}{2}, \frac{r}{2n}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), r\right) \tag{3.6}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^k x$ in (3.6), we obtain

$$P_{\mu,\nu}\left(f(2^k x) - \frac{f(2^{k+1}x)}{2}, \frac{r}{2n}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(2^k x, 2^k x, 0, \cdots, 0\right), r\right) \tag{3.7}$$

for all $x \in X$ and all $r > 0$. Using (3.1), $(IFN3)$ in (3.7), we arrive

$$P_{\mu,\nu}\left(f(2^k x) - \frac{f(2^{k+1}x)}{2}, \frac{r}{2n}\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), \frac{r}{b^k}\right) \tag{3.8}$$

for all $x \in X$ and all $r > 0$. It is easy to verify from (3.8), that

$$P_{\mu,\nu}\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{(k+1)}},\frac{r}{2^{k+1}\cdot n}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),\frac{r}{b^k}\right) \tag{3.9}$$

holds for all $x\in X$ and all $r>0$. Replacing r by $b^n r$ in (3.9), we get

$$P_{\mu,\nu}\left(\frac{f(2^k x)}{2^k}-\frac{f(2^{k+1}x)}{2^{(k+1)}},\frac{b^k r}{2^{k+1}\cdot n}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),r\right) \tag{3.10}$$

for all $x\in X$ and all $r>0$. It is easy to see that

$$f(x)-\frac{f(2^k x)}{2^k}=\sum_{i=0}^{k-1}\frac{f(2^i x)}{2^i}-\frac{f(2^{i+1}x)}{2^{(i+1)}} \tag{3.11}$$

for all $x\in X$. From equations (3.10) and (3.11), we have

$$\begin{aligned}P_{\mu,\nu}\left(f(x)-\frac{f(2^k x)}{2^k},\sum_{i=0}^{k-1}\frac{b^i r}{2^i\cdot 2n}\right)&\geq_{L^*} M_{i=0}^{k-1}\left\{P_{\mu,\nu}\left(\frac{f(2^i x)}{2^i}-\frac{f(2^{i+1}x)}{2^{(i+1)}},\frac{b^i r}{2^i\cdot 2n}\right)\right\}\\&\geq_{L^*} M_{i=0}^{k-1}\left\{P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),r\right)\right\}\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),r\right)\end{aligned} \tag{3.12}$$

for all $x\in X$ and all $r>0$. Replacing x by $2^m x$ in (3.12) and using (3.1), $(IFN3)$, we obtain

$$P_{\mu,\nu}\left(\frac{f(2^m x)}{2^m}-\frac{f(2^{k+m}x)}{2^{(k+m)}},\sum_{i=0}^{k-1}\frac{b^i r}{2^{(i+m)}\cdot 2n}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),\frac{r}{b^m}\right) \tag{3.13}$$

for all $x\in X$ and all $r>0$ and all $m,k\geq 0$. Replacing r by $b^m r$ in (3.13), we get

$$P_{\mu,\nu}\left(\frac{f(2^m x)}{2^m}-\frac{f(2^{k+m}x)}{2^{(k+m)}},\sum_{i=m}^{m+k-1}\frac{b^i r}{2^i\cdot 2n}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),r\right) \tag{3.14}$$

for all $x\in X$ and all $r>0$ and all $m,k\geq 0$. It follows from (3.14) that

$$P_{\mu,\nu}\left(\frac{f(2^m x)}{2^m}-\frac{f(2^{k+m}x)}{2^{(k+m)}},r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),\frac{r}{\sum_{i=m}^{m+k-1}\frac{b^i}{2^i\cdot 2n}}\right) \tag{3.15}$$

for all $x\in X$ and all $r>0$ and all $m,k\geq 0$. Since $0<b<2$ and $\sum_{i=0}^{k}\left(\frac{b}{2}\right)^i<\infty$, this implies $\left\{\frac{f(2^k x)}{2^k}\right\}$ is a Cauchy sequence in $(Y,P'_{\mu,\nu},M)$. Since $(Y,P'_{\mu,\nu},M)$ is a complete IFN space, this sequence converges to some point $A(x)\in Y$. So one can define the mapping $A:X\to Y$ by

$$P_{\mu,\nu}\left(A(x)-\frac{f(2^k x)}{2^k},r\right)\to 1_{L^*}\quad as\quad k\to\infty, r>0 \tag{3.16}$$

for all $x\in X$. Letting $m=0$ in (3.15), we get

$$P_{\mu,\nu}\left(f(x)-\frac{f(2^k x)}{2^k},r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),\frac{r}{\sum_{i=0}^{k-1}\frac{b^i}{2^i\cdot 2n}}\right) \tag{3.17}$$

for all $x\in X$ and all $r>0$. Letting $k\to\infty$ in (3.17), we arrive

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x,x,0,\cdots,0\right),rn(2-b)\right)$$

for all $x\in X$ and all $r>0$. To prove A satisfies the additive functional equation (1.1), replacing $(x,y_1,y_2,\cdots,y_n)$ by $\left(2^k x,2^k y_1,2^k y_2,...,2^k y_n\right)$ and dividing by 2^k in (3.2), we obtain

$$P_{\mu,\nu}\left(\frac{1}{2^k}Df\left(2^k x,2^k y_1,\cdots,2^k y_n\right),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(2^k x,2^k y_1,\cdots,2^k y_n\right),2^k r\right) \tag{3.18}$$

for all $x,y_1,y_2,...,y_n\in X$ and all $r>0$. Now,

$$P_{\mu,\nu}\left(A(x)-\sum_{l=1}^{n}\left(\frac{A(x+ly_l)+A(x-ly_l)}{2n}\right),r\right)$$

$$\geq_{L^*} M\left\{P_{\mu,\nu}\left(A(x)-\frac{f(2^k x)}{2^k},\frac{r}{3}\right),P_{\mu,\nu}\left(-\sum_{l=1}^{n}\left(\frac{A(x+ly_l)+A(x-ly_l)}{2n}\right)+\frac{1}{2^k}\sum_{l=1}^{n}\left(\frac{f(2^k(x+ly_l))+f(2^k(x-ly_l))}{2n}\right),\frac{r}{3}\right),\right.$$
$$\left. P_{\mu,\nu}\left(\frac{f(2^k x)}{2^k}-\frac{1}{2^k}\sum_{l=1}^{n}\left(\frac{f(2^k(x+ly_l))+f(2^k(x-ly_l))}{2n}\right),\frac{r}{3}\right)\right\} \qquad (3.19)$$

for all $x, y_1, y_2, ..., y_n \in X$ and all $r>0$. Using (3.16), (3.18), (3.2) and $(IFN2)$ in (3.19), we arrive

$$A(x)=\sum_{l=1}^{n}\left(\frac{A(x+ly_l)+A(x-ly_l)}{2n}\right)$$

for all $x, y_1, y_2, ..., y_n \in X$. Hence A satisfies the additive functional equation (1.1). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive functional mapping satisfying (3.4) and (3.5). Hence,

$$P_{\mu,\nu}(A(x)-A'(x),r)\geq_{L^*} M\left\{P_{\mu,\nu}\left(\frac{A(2^k x)}{2^k}-\frac{f(2^k x)}{2^k},\frac{r}{2}\right),P_{\mu,\nu}\left(\frac{A'(2^k x)}{2^k}-\frac{A(2^k x)}{2^{3k}},\frac{r}{2}\right)\right\}$$
$$\geq_{L^*} P'_{\mu,\nu}\left(\xi(2^k x,2^k x,0,\cdots,0),\frac{r\,2^k\,n(2-b)}{2}\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi(x,x,0,\cdots,0),\frac{r\,2^k\,n(2-b)}{2b^k}\right)$$

for all $x\in X$ and all $r>0$. Since $\lim_{k\to\infty}\frac{r\,2^k\,n(2-b)}{2b^k}=\infty$, we obtain $\lim_{k\to\infty}P'_{\mu,\nu}\left(\xi(x,x,0,\cdots,0),\frac{r\,2^k\,n(2-b)}{2b^k}\right)=1_{L^*}$. Thus $P_{\mu,\nu}(A(x)-A'(x),r)=1_{L^*}$ for all $x\in X$ and all $r>0$, hence $A(x)=A'(x)$. Therefore $A(x)$ is unique.

For $\delta=-1$, we can prove the result by a similar method. This completes the proof of the theorem.

From Theorem 3.1, we obtain the following corollary concerning the stability for the functional equation (1.1).

Corollary 3.2 Suppose that a function $f: X\to Y$ satisfies the inequality

$$P_{\mu,\nu}(Df(x,y_1,\cdots y_n),r)\geq_{L^*}\begin{cases}P'_{\mu,\nu}(\lambda,r), & \\ P'_{\mu,\nu}\left(\lambda\left\{\|x\|^s+\sum_{l=1}^{n}\|y_l\|^s\right\},r\right), & s\neq 1\\ P'_{\mu,\nu}\left(\lambda\left\{\|x\|^s\prod_{l=1}^{n}\|y_l\|^s+\left\{\|x\|^{(n+1)s}+\sum_{l=1}^{n}\|y_l\|^{(n+1)s}\right\}\right\},r\right), & s\neq\frac{1}{n+1}\end{cases} \qquad (3.20)$$

for all all $x, y_1, y_2, ..., y_n \in X$ and all $r>0$, where λ, s are constants with $\lambda>0$. Then there exists a unique additive mapping $A: X\to Y$ such that

$$P_{\mu,\nu}(f(x)-A(x),r)\geq_{L^*}\begin{cases}P'_{\mu,\nu}(\lambda,nr),\\ P'_{\mu,\nu}\left(2\lambda\|x\|^s,rn\left|2-2^s\right|\right),\\ P'_{\mu,\nu}\left(2\lambda\|x\|^{(n+1)s},rn\left|2-2^{(n+1)s}\right|\right)\end{cases} \qquad (3.21)$$

for all $x\in X$ and all $r>0$.

4. INTUITIONISTIC FUZZY NORMED STABILITY: FIXED POINT METHOD

In this section, using the fixed point alternative approach, we prove the generalized Ulam - Hyers stability of the functional equation (1.1) in intuitionistic fuzzy normed spaces. Throughout this section, let us consider X, $(Z, P_{\mu,\nu}, M)$ and $(Y, P'_{\mu,\nu}, M)$ are linear space, Intuitionistic fuzzy normed space and Complete Intuitionistic fuzzy normed space.

For to prove the stability result we define the following: a_i is a constant such that

$$a_i = \begin{cases} 2 & if \quad i = 0, \\ \dfrac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that

$$\Omega = \{ g \mid g : X \to Y, g(0) = 0 \}.$$

Theorem 4.1. Let $f : X \to Y$ be a mapping for which there exist a function $\xi : X^{n+1} \to Z$ with the condition

$$\lim_{k\to\infty} P'_{\mu,\nu}\left(\xi\left(a_i^k x, a_i^k y_1, \cdots, a_i^k y_n\right), a_i^k r\right) = 1_{L^*}, \ \forall\, x, y_1, \cdots, y_n \in X,\ r > 0 \tag{4.1}$$

and satisfying the functional inequality

$$P_{\mu,\nu}\left(Df\left(x, y_1, y_2, \cdots, y_n\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, y_1, y_2, \cdots, y_n\right), r\right),\ \forall\, x, y_1, y_2, \cdots, y_n \in X,\ r > 0. \tag{4.2}$$

If there exists $L = L(i)$ such that the function $x \to \gamma(x) = \xi\left(\frac{x}{2}, \frac{x}{2}, 0, \cdots, 0\right)$, has the property

$$P'_{\mu,\nu}\left(L\frac{\gamma(a_i x)}{a_i}, r\right) = P'_{\mu,\nu}\left(\gamma(x), r\right), \quad \forall\, x \in X,\ r > 0. \tag{4.3}$$

Then there exists unique additive function $A : X \to Y$ satisfying the functional equation (1.1) and

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\left(\frac{L^{1-i}}{1-L}\right)\gamma(x), nr\right),\ \forall\, x \in X, r > 0. \tag{4.4}$$

Proof. Let d be a general metric on Ω, such that

$$d(g,h) = inf\left\{K \in (0,\infty) \mid P_{\mu,\nu}\left(g(x) - h(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(K\gamma(x), r\right), x \in X,\ r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{a_i} g(a_i x)$, for all $x \in X$. Now for all $g, h \in \Omega$,

$$\begin{aligned} d(g,h) \leq K &\Rightarrow P_{\mu,\nu}\left(g(x) - h(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(K\gamma(x), r\right),\ x \in X, \\ &\Rightarrow P_{\mu,\nu}\left(\frac{1}{a_i} g(a_i x) - \frac{1}{a_i} h(a_i x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(K\gamma(a_i x), a_i r\right),\ x \in X, \\ &\Rightarrow P_{\mu,\nu}\left(\frac{1}{a_i} g(a_i x) - \frac{1}{a_i} h(a_i x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\frac{K}{a_i}\gamma(a_i x), r\right),\ x \in X, \\ &\Rightarrow P_{\mu,\nu}\left(Tg(x) - Th(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(KL\gamma(x), r\right),\ x \in X, \\ &\Rightarrow d\left(Tg, Th\right) \leq LK. \end{aligned}$$

This gives $d(Tg, Th) \leq Ld(g,h)$, for all $g, h \in \Omega$, i.e., T is a strictly contractive mapping of Ω with Lipschitz constant $L = \frac{1}{a_i}$. Replacing $(x, y_1, y_2, \cdots, y_n)$ by $(x, x, 0, \cdots, 0)$ in (4.2), we get

$$P_{\mu,\nu}\left(2nf(x) - nf(2x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), r\right),\ \forall \ \ x \in X, r > 0. \tag{4.5}$$

Using $(IFN3)$ in (4.5), we arrive

$$P_{\mu,\nu}\left(f(x) - \frac{f(2x)}{2}, r\right) \geq_{L^*} P'_{\mu,\nu}\left(\xi\left(x, x, 0, \cdots, 0\right), 2nr\right),\ \forall \ \ x \in X, r > 0. \tag{4.6}$$

With the help of (4.3), when $i = 0$, it follows from (4.6), that

$$P_{\mu,\nu}\left(f(x) - \frac{f(2x)}{2}, r\right) \geq_{L^*} P'_{\mu,\nu}\left(\gamma(x), 2nr\right),\ \forall \ \ x \in X, r > 0.$$

$$\Rightarrow d\left(f, Tf\right) \leq L = L^1 = L^{1-i} < \infty. \tag{4.7}$$

Replacing x by $\frac{x}{2}$ in (4.5), we obtain

$$P_{\mu,\nu}\left(2f\left(\frac{x}{2}\right)-f(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(\frac{x}{2},\frac{x}{2},0,\cdots,0\right),nr\right),\quad \forall\quad x\in X, r>0. \tag{4.8}$$

With the help of (4.3), when $i=1$, it follows from (4.8), that

$$P_{\mu,\nu}\left(2f\left(\frac{x}{2}\right)-f(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\gamma(x),nr\right),\quad \forall\quad x\in X, r>0,$$

$$\Rightarrow d\left(Tf,f\right)\leq 1=L^0=L^{1-i}. \tag{4.9}$$

Then from (4.7) and (4.9), we can conclude

$$d\left(f,Tf\right)\leq L^{1-i}<\infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x)\xrightarrow{P_{\mu,\nu}}\frac{f\left(a_i^k x\right)}{a_i^k},\ k\to\infty,\ \forall\ x\in X. \tag{4.10}$$

Replacing $(x,y_1,\cdots,y_n)$ by $\left(a_i^k x, a_i^k y_1,\cdots,a_i^k y_n\right)$ in (4.2), we arrive

$$P_{\mu,\nu}\left(\frac{1}{a_i^k}Df\left(a_i^k x,a_i^k y_1,\cdots,a_i^k y_n\right),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\xi\left(a_i^k x,a_i^k y_1,\cdots,a_i^k y_n\right),a_i^k r\right),\quad \forall\quad x_1,\cdots,x_n\in X, r>0. \tag{4.11}$$

In order to prove A satisfies (1.1), the proof is similar to that of Theorem 3.1. Using fixed point alternative, A is the uniquie fixed point in T the set

$$\mathrm{B}=\left\{h\in\Omega \mid d\left(f,A\right)<\infty\right\},$$

such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(K\gamma(x),r\right),\quad \forall\quad x\in X, r>0. \tag{4.13}$$

Again using the fixed point alternative, we obtain

$$d\left(f,A\right)\leq\frac{1}{1-L}d\left(f,Tf\right)\Rightarrow d\left(f,A\right)\leq\frac{L^{1-i}}{1-L}.$$

Hence, we have

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\left(\frac{L^{1-i}}{1-L}\right)\gamma(x),nr\right),\ \forall\ x\in X, r>0. \tag{4.14}$$

This completes the proof of the theorem.

From Theorem 4.1, we obtain the following corollary concerning the stability for the functional equation (1.1).

Corollary 4.2 Suppose that a function $f:X\to Y$ satisfies the inequality

$$P_{\mu,\nu}\left(Df\left(x,y_1,\cdots y_n\right),r\right)\geq_{L^*}\begin{cases}P'_{\mu,\nu}\left(\lambda,r\right), & \\ P'_{\mu,\nu}\left(\lambda\left\{\|x\|^s+\sum_{l=1}^{n}\|y_l\|^s\right\},r\right), & s\neq 1\\ P'_{\mu,\nu}\left(\lambda\left\{\|x\|^s\prod_{l=1}^{n}\|y_l\|^s+\left\{\|x\|^{(n+1)s}+\sum_{l=1}^{n}\|y_l\|^{(n+1)s}\right\}\right\},r\right), & s\neq\frac{1}{n+1}\end{cases} \tag{4.15}$$

for all all $x,y_1,y_2,...,y_n\in X$ and all $r>0$, where λ,s are constants with $\lambda>0$. Then there exists a unique quadratic mapping $A:X\to Y$ such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*}\begin{cases}P'_{\mu,\nu}\left(\lambda,|n|r\right)\\ P'_{\mu,\nu}\left(\dfrac{2\lambda}{n\left|2-2^s\right|}\|x\|^s,r\right)\\ P'_{\mu,\nu}\left(\dfrac{2\lambda}{n\left|2-2^{(n+1)s}\right|}\|x\|^{(n+1)s},r\right)\end{cases}\tag{4.16}$$

for all $x\in X$ and all $r>0$.

Proof.

Setting $$\xi\left(x,y_1,y_2,\cdots,y_n\right)=\begin{cases}\lambda\\ \lambda\left(\|x\|^s+\sum_{i=1}^{n}\|y_i\|^s\right)\\ \lambda\left(\|x\|^s\prod_{i=1}^{n}\|y_i\|^s+\|x_i\|^{(n+1)s}+\sum_{i=1}^{n}\|y_i\|^{(n+1)s}\right)\end{cases}$$

for all $x,y_1,y_2,...,y_n\in X$. Then

$$P'_{\mu,\nu}\left(\frac{1}{a_i^k}\xi\left(a_i^kx,a_i^ky_1,a_i^ky_2,\cdots,a_i^ky_n\right),r\right)=\begin{cases}P'_{\mu,\nu}\left(\dfrac{\rho}{a_i^k},r\right)\\ P'_{\mu,\nu}\left(\dfrac{\rho}{a_i^k}\left(\|a_i^kx\|^s+\sum_{i=1}^{n}\|a_i^ky_i\|^s\right),r\right)\\ P'_{\mu,\nu}\left(\dfrac{\rho}{a_i^k}\left(\|a_i^kx\|^s\prod_{i=1}^{n}\|a_i^ky_i\|^s+\|a_i^kx\|^{(n+1)s}+\sum_{i=1}^{n}\|a_i^ky_i\|^{(n+1)s}\right),r\right)\end{cases}$$

$$=\begin{cases}P'_{\mu,\nu}\left(a_i^{-k}\lambda,r\right)\\ P'_{\mu,\nu}\left(a_i^{(s-1)k}\lambda\left(\|x\|^s+\sum_{i=1}^{n}\|y_i\|^s\right),r\right)\\ P'_{\mu,\nu}\left(a_i^{((n+1)s-1)k}\left(\|x\|^s\prod_{i=1}^{n}\|y_i\|^s+\|x\|^{(n+1)s}+\sum_{i=1}^{n}\|y_i\|^{(n+1)s}\right),r\right)\end{cases}=\begin{cases}\to 1_{L^*}\ \ as\ \ k\to\infty,\\ \to 1_{L^*}\ \ as\ \ k\to\infty,\\ \to 1_{L^*}\ \ as\ \ k\to\infty.\end{cases}$$

i.e., (3.1) is holds. But we have $\gamma(x)=\xi\left(\frac{x}{2},\frac{x}{2},0,\ldots,0\right)$ has the property $\gamma(x)\leq L\cdot\frac{1}{a_i}\gamma\left(a_ix\right)$ for all $x\in X$. Hence

$$P'_{\mu,\nu}\left(\gamma(x),r\right)=P'_{\mu,\nu}\left(\xi\left(\tfrac{x}{2},\tfrac{x}{2},0,\cdots0\right),r\right)=\begin{cases}P'_{\mu,\nu}\left(\lambda,nr\right)\\ P'_{\mu,\nu}\left(\lambda 2^{1-s}\|x\|^s,nr\right)\\ P'_{\mu,\nu}\left(\lambda 2^{1-(n+1)s}\|x\|^{(n+1)s},nr\right)\end{cases}$$

Now,

$$P'_{\mu,\nu}\left(\frac{1}{a_i}\gamma(a_ix),r\right)=\begin{cases}P'_{\mu,\nu}\left(a_i^{-1}\lambda,nr\right)\\ P'_{\mu,\nu}\left(\lambda a_i^{1-s}2^{1-s}\|x\|^s,nr\right)\\ P'_{\mu,\nu}\left(\lambda a_i^{1-(n+1)s}2^{1-(n+1)s}\|x\|^{(n+1)s},nr\right)\end{cases}$$

for all $x\in X$. Hence the inequality (3.3) holds either, $L=2^{1-s}$ for $s>1$ if $i=0$ and $L=2^{s-1}$ for $s<1$ if $i=1$. From (5.4),

Case:1 $L = 2^{1-s}$ for $s > 1$ if $i = 0$,.

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\left(\frac{2^{1-s}}{1-2^{1-s}}\right)\gamma(x), nr\right) = P'_{\mu,\nu}\left(\frac{2\lambda}{n\left|2^s - 2\right|}\|x\|^s, r\right).$$

Case:2 $L = 2^{s-1}$ for $s < 1$ if $i = 1$,

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\left(\frac{1}{1-2^{s-1}}\right)\gamma(x), nr\right) = P'_{\mu,\nu}\left(\frac{2\lambda}{n\left|2 - 2^s\right|}\|x\|^s, r\right).$$

Similarly, the inequality (4.3) holds either, $L = 2^{-1}$ if $i = 0$ and $L = 2$ if $i = 1$ for condition (i) and the inequality (4.3) holds either, $L = 2^{1-(n+1)s}$ for $s > \frac{1}{n+1}$ if $i = 0$ and $L = 2^{(n+1)s-1}$ for $s < \frac{1}{n+1}$ if $i = 1$ for condition (iii). Hence the proof is complete.

5. NON-ARCHIMEDEAN FUZZY $\varphi - 2 -$ NORMED STABILITY: DIRECT METHOD

In this section, the generalized Ulam - Hyers stability of the additive functional equation (1.2) in non-Archimedean fuzzy $\varphi - 2 -$ normed space is provided. Here after, throughout this section, assume that R be a non-Archimedean field, X be vector space over R , $(Y, N', \Diamond)$ be a non-Archimedean fuzzy $\varphi - 2 -$ Banach space over R and $(Z, N', \Diamond)$ be an non-Archimedean fuzzy $\varphi - 2 -$ normed space.

Theorem 5.1 *Let $\gamma \in \{-1,1\}$ be fixed and let $\alpha : X^{n+1} \to Z$ be a mapping such that for some τ with* $0 < \left(\frac{\varphi(\tau)}{\varphi\left(2/n\right)}\right)^{\gamma} < 1$

$$N'\left(\alpha\left(\left(2/n\right)^{\gamma} x, 0, 0, ..., 0\right), a, r\right) \geq N'\left(\tau^{\gamma}\alpha\left(x, 0, 0, ..., 0\right), a, r\right) \tag{5.1}$$

for all $x, a \in X$ and all $r > 0$, and

$$\lim_{k\to\infty} N'\left(\alpha\left(\left(n/2\right)^{\gamma k} x, \left(n/2\right)^{\gamma k} y_1, \left(n/2\right)^{\gamma k} y_2, ..., \left(n/2\right)^{\gamma k} y_n, a, \frac{r}{\left[\varphi\left(\left(n/2\right)^k\right)\right]^{\gamma}}\right)\right) = 1 \tag{5.2}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(DF(x, y_1, y_2, \cdots, y_n), a, r\right) \geq N'\left(\alpha(x, y_1, y_2, \cdots, y_n), a, r\right) \tag{5.3}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Then the limit

$$R(x) = N - \lim_{k\to\infty}\left(n/2\right)^{\gamma k} f\left(\left(n/2\right)^{\gamma k} x\right) \tag{5.4}$$

exists for all $x \in X$ and the mapping $R : X \to Y$ is a unique reciprocal mapping satisfying (1.2) and

$$N\left(f(x) - R(x), a, r\right) \geq N'\left(\alpha\left(\left(n/2\right)x, 0, ..., 0\right), a, r\left|\varphi\left(2/n\right) - \varphi(\kappa)\right|\right) \tag{5.5}$$

for all $x, a \in X$ and all $r > 0$.

Proof. First assume $\gamma = 1$. $(x, y_1, y_2, \cdots, y_n)$ by $(x, 0, 0, \cdots, 0)$ in (5.3), we get

$$N\left(f\left(\left(2/n\right)x\right) - \left(n/2\right) f(x), a, r\right) \geq N'\left(\alpha\left(x, 0, 0, ..., 0\right), a, r\right) \tag{5.6}$$

for all $x, a \in X$ and all $r > 0$. Replacing x by $\left(\frac{n}{2}\right)x$ in (5.6), we get

$$N\left(f(x)-\left(\frac{n}{2}\right)f\left(\left(\frac{n}{2}\right)x\right),a,r\right)\geq N'\left(\alpha\left(\left(\frac{n}{2}\right)x,0,0,...,0\right),a,r\right) \tag{5.7}$$

for all $x, a \in X$ and all $r > 0$. Replacing x by $\left(\frac{n}{2}\right)^k x$ in (5.7), we obtain

$$N\left(f\left(\left(\frac{n}{2}\right)^k x\right)-\left(\frac{n}{2}\right)f\left(\left(\frac{n}{2}\right)^{k+1}x\right),a,r\right)\geq N'\left(\alpha\left(\left(\frac{n}{2}\right)^{k+1}x,0,0,...,0\right),a,r\right) \tag{5.8}$$

for all $x, a \in X$ and all $r > 0$. Using (5.1), $(NAF6)$ in (5.8), we arrive

$$N\left(\left(\frac{n}{2}\right)^k f\left(\left(\frac{n}{2}\right)^k x\right)-\left(\frac{n}{2}\right)^{k+1} f\left(\left(\frac{n}{2}\right)^{k+1}x\right),a,\frac{r}{\varphi\left(\left(\frac{2}{n}\right)^k\right)}\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,\frac{r}{\varphi\left(\tau^{k+1}\right)}\right) \tag{5.9}$$

for all $x, a \in X$ and all $r > 0$. Replacing r by $\varphi\left(\tau^{k+1}\right)r$ in (5.9), we get

$$N\left(\left(\frac{n}{2}\right)^k f\left(\left(\frac{n}{2}\right)^k x\right)-\left(\frac{n}{2}\right)^{k+1} f\left(\left(\frac{n}{2}\right)^{k+1}x\right),a,\frac{\varphi\left(\tau^{k+1}\right)r}{\varphi\left(\left(\frac{2}{n}\right)^k\right)}\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,r\right) \tag{5.10}$$

for all $x, a \in X$ and all $r > 0$. It is easy to verify that

$$f(x)-\left(\frac{n}{2}\right)^i f\left(\left(\frac{n}{2}\right)^i x\right)=\sum_{i=0}^{k-1}\left(\left(\frac{n}{2}\right)^i f\left(\left(\frac{n}{2}\right)^i x\right)-\left(\frac{n}{2}\right)^{i+1} f\left(\left(\frac{n}{2}\right)^{i+1}x\right)\right) \tag{5.11}$$

for all $x \in X$. From equations (5.10) and (5.11), we have

$$N\left(f(x)-\left(\frac{n}{2}\right)^k f\left(\left(\frac{n}{2}\right)^k x\right),a,\sum_{i=0}^{k-1}\frac{[\varphi(\tau)]^{i+1}r}{\left[\varphi\left(\frac{2}{n}\right)\right]^i}\right)\geq min\bigcup_{i=0}^{k-1}\left\{N\left(\left(\frac{n}{2}\right)^i f\left(\left(\frac{n}{2}\right)^i x\right)-\left(\frac{n}{2}\right)^{i+1} f\left(\left(\frac{n}{2}\right)^{i+1}x\right),a,\frac{[\varphi(\tau)]^{i+1}r}{\left[\varphi\left(\frac{2}{n}\right)\right]^i}\right)\right\}$$

$$\geq min\bigcup_{i=0}^{n-1}\left\{N'\left(\alpha\left(x,0,0,...,0\right),a,r\right)\right\}\geq N'\left(\alpha\left(x,0,0,...,0\right),a,r\right) \tag{5.12}$$

for all $x, a \in X$ and all $r > 0$. Replacing x by $\left(\frac{n}{2}\right)^m x$ in (5.12) and using (5.1), $(NAF6)$, we obtain

$$N\left(\left(\frac{n}{2}\right)^m f\left(\left(\frac{n}{2}\right)^m x\right)-\left(\frac{n}{2}\right)^{k+m} f\left(\left(\frac{n}{2}\right)^{k+m}x\right),a,\sum_{i=0}^{k-1}\frac{[\varphi(\tau)]^{i+1}r}{\left[\varphi\left(\frac{2}{n}\right)\right]^{i+m}}\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,\frac{r}{[\varphi(\tau)]^m}\right) \tag{5.13}$$

for all $x, a \in X$ and all $r > 0$ and all $m, k \geq 0$. Replacing r by $\varphi(\tau^m)r$ in (5.13), we get

$$N\left(\left(\frac{n}{2}\right)^m f\left(\left(\frac{n}{2}\right)^m x\right)-\left(\frac{n}{2}\right)^{k+m} f\left(\left(\frac{n}{2}\right)^{k+m}x\right),a,\sum_{i=m}^{m+k+1}\frac{[\varphi(\tau)]^{i+1}r}{\left[\varphi\left(\frac{2}{n}\right)\right]^i}\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,r\right) \tag{5.14}$$

for all $x, a \in X$ and all $r > 0$ and all $m, k \geq 0$. It follows from (5.14) that

$$N\left(\left(\frac{n}{2}\right)^m f\left(\left(\frac{n}{2}\right)^m x\right)-\left(\frac{n}{2}\right)^{k+m} f\left(\left(\frac{n}{2}\right)^{k+m}x\right),a,r\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,\sum_{i=m}^{m+k+1}\frac{[\varphi(\tau)]^{i+1}r}{\left[\varphi\left(\frac{2}{n}\right)\right]^i}\right) \tag{5.15}$$

for all $x, a \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < \tau < \left(2/n\right)$ and $\sum_{i=0}^{n}\left(\frac{\varphi(\tau)}{\varphi\left(2/n\right)}\right)^{\gamma} < \infty$, implies that $\left\{\frac{f\left(\left(n/2\right)^k x\right)}{\left(2/n\right)^k}\right\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a non-Archimedean fuzzy $\varphi-2-$ Banach space, this sequence converges to some point $R(x) \in Y$. So one can define the mapping $R: X \to Y$ by

$$R(x) = N - \lim_{k\to\infty} \frac{f\left(\left(n/2\right)^k x\right)}{\left(2/n\right)^k}$$

for all $x \in X$. Letting $m = 0$ in (5.15), we get

$$N\left(f(x) - \left(n/2\right)^k f\left(\left(n/2\right)^k x\right), a, r\right) \geq N'\left(\alpha(x,0,0,...,0), a, \sum_{i=0}^{k+1}\frac{[\varphi(\tau)]^{i+1} r}{\left[\varphi\left(2/n\right)\right]^i}\right) \tag{5.16}$$

for all $x, a \in X$ and all $r > 0$. Letting $k \to \infty$ in (5.16) and using $(NAF5)$, we arrive

$$N\left(f(x) - R(x), a, r\right) \geq N'\left(\alpha(x,0,0,...,0), a, r\left(\varphi\left(2/n\right) - \varphi(\tau)\right)\right)$$

for all $x, a \in X$ and all $r > 0$. To prove R satisfies (1.2), replacing $(x, y_1, y_2, \cdots, y_n)$ by $\left(\left(n/2\right)^k x, \left(n/2\right)^k y_1, \left(n/2\right)^k y_2, \cdots, \left(n/2\right)^k y_n\right)$ in (5.3), respectively , we obtain

$$N\left(\left(n/2\right)^k DF\left(\left(n/2\right)^k x, \left(n/2\right)^k y_1, \left(n/2\right)^k y_2, \cdots, \left(n/2\right)^k y_n\right), a, r\right) \geq N'\left(\alpha\left(\left(n/2\right)^k x, \left(n/2\right)^k y_1, \left(n/2\right)^k y_2, \cdots, \left(n/2\right)^k y_n\right), a, \varphi\left(2/n\right)^k r\right) \tag{5.17}$$

for all $r > 0$ and all $x, y_1, y_2, ..., y_2, a \in X$. Now,

$$\begin{aligned} N\left(R\left(\left(2/n\right)x\right) - \sum_{l=1}^{n}\left(\frac{R(x+ly_l)R(x-ly_l)}{R(x+ly_l)+R(x-ly_l)}\right), a, r\right) \geq \min\Bigg\{ & N\left(R\left(\left(2/n\right)x\right) - \left(n/2\right) f\left(\left(n/2\right)\left(2/n\right)x\right), a, \frac{r}{3}\right), \\ & N\left(-R\left(\sum_{l=1}^{n}\left(\frac{R(x+ly_l)R(x-ly_l)}{R(x+ly_l)+R(x-ly_l)}\right)\right) + \left(n/2\right) f\left(\sum_{l=1}^{n}\left(\frac{f\left(\left(n/2\right)(x+ly_l)\right) f\left(\left(n/2\right)(x-ly_l)\right)}{f\left(\left(n/2\right)(x+ly_l)\right) + f\left(\left(n/2\right)(x-ly_l)\right)}\right)\right), a, \frac{r}{3}\right), \\ & N\left(\left(n/2\right) f\left(\left(n/2\right)\left(2/n\right)x\right) - \left(n/2\right) f\left(\sum_{l=1}^{n}\left(\frac{f\left(\left(n/2\right)(x+ly_l)\right) f\left(\left(n/2\right)(x-ly_l)\right)}{f\left(\left(n/2\right)(x+ly_l)\right) + f\left(\left(n/2\right)(x-ly_l)\right)}\right)\right), a, \frac{r}{3}\right)\Bigg\} \end{aligned} \tag{5.18}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Using (5.17) and $(NAF5)$ in (5.18), we arrive

$$\begin{aligned} N\left(DR(x, y_1, y_2, \cdots, y_n), a, r\right) &\geq min\left\{1,1,1, N'\left(\alpha\left(\left(n/2\right)^k x, \left(n/2\right)^k y_1, ..., \left(n/2\right)^k y_1\right), a, \varphi\left(\left(2/n\right)^k\right) r\right)\right\} \\ &\geq N'\left(\alpha\left(\left(n/2\right)^k x, \left(n/2\right)^k y_1, ..., \left(n/2\right)^k y_1\right), a, \varphi\left(\left(2/n\right)^k\right) r\right) \end{aligned} \tag{5.19}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Letting $k \to \infty$ in (5.19) and using (5.2), we see that

$$N\left(DR(x, y_1, y_2, \cdots, y_n), a, r\right) = 1$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Using $(NAF2)$ in the above inequality, we get

$$R\left(\frac{2x}{n}\right)=\sum_{l=1}^{n}\left(\frac{R(x+ly_l)R(x-ly_l)}{R(x+ly_l)+R(x-ly_l)}\right)$$

for all $x, y_1, y_2, ..., y_2, a \in X$. Hence R satisfies the reciprocal functional equation (1.2). In order to prove $R(x)$ is unique, let $R'(x)$ be another reciprocal functional equation satisfying (5.4) and (5.5). Hence,

$$N\left(R(x)-R'(x),a,r\right)$$

$$\geq min\left\{N\left(\left(\tfrac{n}{2}\right)^k R\left(\left(\tfrac{n}{2}\right)^k k\right)-\left(\tfrac{n}{2}\right)^k f\left(\left(\tfrac{n}{2}\right)^k k\right),a,\frac{r}{2}\right),N\left(\left(\tfrac{n}{2}\right)^k R'\left(\left(\tfrac{n}{2}\right)^k k\right)-\left(\tfrac{n}{2}\right)^k f\left(\left(\tfrac{n}{2}\right)^k k\right),a,\frac{r}{2}\right)\right\}$$

$$\geq N'\left(\alpha\left(\left(\tfrac{n}{2}\right)^k x,0,0,...,0\right),a,\frac{r\,\varphi\left(\left(\tfrac{2}{n}\right)^k\right)\left(\varphi\left(\tfrac{2}{n}\right)-\varphi(\tau)\right)}{2}\right)\geq N'\left(\alpha\left(x,0,0,...,0\right),a,\frac{r\,\varphi\left(\left(\tfrac{2}{n}\right)^k\right)\left(\varphi\left(\tfrac{2}{n}\right)-\varphi(\tau)\right)}{2\varphi\left(\tau^k\right)}\right)$$

for all $x, a \in X$ and all $r > 0$. Since $\lim_{k\to\infty}\frac{r\,\varphi\left(\left(\tfrac{2}{n}\right)^k\right)\left(\varphi\left(\tfrac{2}{n}\right)-\varphi(\tau)\right)}{2\varphi\left(\tau^k\right)}=\infty$, we obtain

$$\lim_{k\to\infty}N'\left(\alpha\left(x,0,0,...,0\right),a,\frac{r\,\varphi\left(\left(\tfrac{2}{n}\right)^k\right)\left(\varphi\left(\tfrac{2}{n}\right)-\varphi(\tau)\right)}{2\varphi\left(\tau^k\right)}\right)=1.$$

Thus $N\left(R(x)-R'(x),a,r\right)=1$ for all $x, a \in X$ and all $r > 0$, hence $R(x)=R'(x)$. Therefore $R(x)$ is unique. For $\gamma = -1$, we can prove the result by a similar method. This completes the proof of the theorem.

The following Corollary is an immediate consequence of Theorem 5.1 concerning the stabilities of (1.2).

Corollary 5.2 *Suppose that a function* $f : X \to Y$ *satisfies the inequality*

$$N\left(DF\left(x,y_1,y_2,\cdots,y_n\right),a,r\right)\geq\begin{cases}N'\left(\varepsilon,a,r\right),\\ N'\left(\varepsilon\left(\|x\|^s+\sum_{i=1}^{n}\|y_i\|^s\right),a,r\right), & s\neq -1;\\ N'\left(\varepsilon\left(\|x\|^s\prod_{i=1}^{n}\|y_i\|^s+\|x_i\|^{(n+1)s}+\sum_{i=1}^{n}\|y_i\|^{(n+1)s}\right),a,r\right), & s\neq\dfrac{-1}{n+1};\end{cases}\tag{5.20}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$, where ε, s are constants with $\varepsilon > 0$. Then there exists a unique reciprocal mapping $R : X^{n+1} \to Y$ such that

$$N\left(f(x)-R(x),r\right)\geq\begin{cases}N'\left(2\varepsilon,a,r\left|\varphi(2)-\varphi(n)\right|\right),\\ N'\left(2^{s+1}\varepsilon\|x\|^s,a,r\left|\varphi\left(2^{s+1}\right)-\varphi\left(n^{s+1}\right)\right|\right),\\ N'\left(2^{(n+1)s+1}\varepsilon\|x\|^{(n+1)s},r\left|\varphi\left(2^{(n+1)s+1}\right)-\varphi\left(n^{(n+1)s+1}\right)\right|\right),\end{cases}\tag{5.21}$$

for all $x, a \in X$ and all $r > 0$.

6. NON-ARCHIMEDEAN FUZZY $\varphi-2-$ NORMED STABILITY: FIXED POINT METHOD

In this section, using the fixed point alternative approach, we prove the generalized Ulam-Hyers stability of the functional equation (1.2) in fuzzy $\varphi-2-$ normed spaces. Throughout this section, assume that R be a non-Archimedean field, X be vector space over R , $(Y, N', \Diamond)$ be a non-Archimedean fuzzy $\varphi-2-$ Banach space over R and $(Z, N', \Diamond)$ be a non-Archimedean fuzzy $\varphi-2-$ normed space.

Theorem 6.1 Let $f: X \to Y$ be a mapping for which there exist a function with the condition $\alpha: X^{n+1} \to Z$ with the condition

$$\lim_{k\to\infty} N'\left(\alpha\left(v_i^k x, v_i^k y_1, \cdots, v_i^k y_n\right), a, \frac{r}{\varphi\left(v_i^k\right)}\right) = 1 \tag{6.1}$$

Where $v_i = \frac{2}{n}$ if $i = 0$ and $v_i = \frac{n}{2}$ if $i = 1$ such that the functional inequality

$$N\left(DF(x, y_1, y_2, \cdots, y_n), a, r\right) \geq N'\left(\alpha\left(x, y_1, y_2, \cdots, y_n\right), a, r\right) \tag{6.2}$$

for all $x, y_1, y_2, \cdots, y_n, a \in X$ and all $r > 0$. If there exists L such that the function

$$x \to \beta(x) = \alpha\left(\frac{nx}{2}, 0, 0, \cdots, 0\right),$$

has the property

$$N'\left(\beta(x), a, r\right) = N'\left(L \cdot v_i\, \beta(v_i x), a, r\right). \tag{6.3}$$

for all $x, a \in X$ and all $r > 0$. Then there exists a unique reciprocal mapping $R: X \to Y$ satisfying the functional equation (1.2) and

$$N\left(R(x) - f(x), a, r\right) \geq N'\left(\left(\frac{L^{1-i}}{1-L}\right)\beta(x), a, r\right) \tag{6.4}$$

for all $x, a \in X$ and all $r > 0$.

Proof. Consider the set $\Omega = \left\{g / g: X \to Y, g(0) = 0\right\}$ and introduce the generalized metric on Ω,

$$d(g,h) = \inf\left\{K \in (0,\infty) / N\left(g(x) - h(x), a, r\right) \geq N'\left(K\beta(x), a, r\right), x \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T: \Omega \to \Omega$ by $Tg(x) = v_i g(v_i x)$, for all $x \in X$. One can show that $d(Tg, Th) \leq L d(g,h)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant $L = v_i$.

Replacing $(x, y_1, y_2, \cdots, y_n)$ by $(x, 0, 0, \cdots, 0)$ in (6.2), we get,

$$N\left(f\left(\left(2/n\right)x\right) - \left(n/2\right) f(x), a, r\right) \geq N'\left(\alpha\left(x, 0, 0, \cdots, 0\right), a, r\right)$$

$$\text{i.e., } N\left(\left(2/n\right) f\left(\left(2/n\right)x\right) - f(x), a, \frac{r}{\varphi\left(n/2\right)}\right) \geq N'\left(\alpha\left(x, 0, 0, \cdots, 0\right), a, r\right) \tag{6.5}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Using (6.3) for the case $i = 0$ it reduces to

$$N\left(\left(2/n\right) f\left(\left(2/n\right)x\right) - f(x), a, \frac{r}{\varphi\left(n/2\right)}\right) \geq N'\left(\beta(x), a, r\right) \quad \text{for all } x, a \in X,\ r > 0.$$

$$\text{i.e., } d\left(Tf, f\right) \leq L \Rightarrow d\left(f, Tf\right) \leq L = L^1 < \infty.$$

Again replacing x by $\left(n/2\right)x$, in (6.5), we get

$$N\left(f(x) - \left(n/2\right) f\left(\left(n/2\right)x\right), a, r\right) \geq N'\left(\alpha\left(\frac{nx}{2}, 0, 0, \cdots, 0\right), a, r\right) \quad \text{for all } x, a \in X,\ r > 0. \tag{6.6}$$

Using (6.3) for the case $i = 1$ it reduces to

$$N\left(f(x) - \left(n/2\right) f\left(\left(n/2\right)x\right), a, r\right) \geq N'\left(\beta(x), a, r\right) \quad \text{for all } x, a \in X,\ r > 0.$$

$$\text{i.e., } d\left(f, Tf\right) \leq 1 \Rightarrow d\left(f, Tf\right) \leq 1 = L^0 < \infty.$$

In both cases, we have

$$d\left(f, Tf\right) \leq L^{1-i}. \tag{6.7}$$

Therefore (A i) holds. By (A ii), it follows that there exists a fixed point R of T in Ω such that

$$R(x) = N - \lim_{k\to\infty} v_i^k f\left(v_i^k x\right) \tag{6.8}$$

To prove that R satisfies (1.2), replacing $(x, y_1, y_2, \cdots, y_n)$ by $(v_i^k x, v_i^k y_1, \cdots, v_i^k y_n)$ in (6.2), we obtain

$$N\left(v_i^k Df\left(v_i^k x, v_i^k y_1, v_i^k y_2, \cdots, v_i^k y_n\right), a, r\right) \geq N'\left(\alpha\left(v_i^k x, v_i^k y_1, v_i^k y_2, \cdots, v_i^k y_n\right), a, \frac{r}{\varphi\left(v_i^k\right)}\right)$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$. Letting $k \to \infty$ in the above inequality and using the definition of $R(x)$, we see that R satisfies (1.2) for all $x, y_1, y_2, \cdots, y_n \in X$. Therefore the mapping R is reciprocal.

By (A iii), since R is the unique fixed point of T in the set $\Delta = \left\{f \in \Omega : d(f, R) < \infty\right\}$, R is the unique function such that

$$N\left(f(x) - R(x), a, r\right) \geq N'\left(K\beta(x), a, r\right)$$

for all $x, a \in X$ and all $r > 0, K > 0$. Again by (A iv), we obtain $d(f, R) \leq \frac{1}{1-L} d(f, Tf)$ this implies $d(f, R) \leq \frac{L^{1-i}}{1-L}$ which yields $N\left(f(x) - R(x), a, r\right) \geq N'\left(\left(\frac{L^{1-i}}{1-L}\right)\beta(x), a, r\right)$ for all $x, a \in X$ and all $r > 0$. This completes the proof of the theorem.

From Theorem 6.1, we obtain the following corollary concerning the stability of (1.2).

Corollary 6.2 Suppose that a function $f : X \to Y$ satisfies the inequality

$$N\left(Df\left(x, y_1, y_2, \cdots, y_n\right), a, r\right) \geq \begin{cases} N'(\lambda, a, r), \\ N'\left(\lambda\left(\|x\|^s + \sum_{i=1}^{n} \|y_i\|^s\right), a, r\right), & s \neq -1; \\ N'\left(\lambda\left(\|x\|^s \prod_{i=1}^{n} \|y_i\|^s + \|x_i\|^{(n+1)s} + \sum_{i=1}^{n} \|y_i\|^{(n+1)s}\right), a, r\right), & s \neq \frac{-1}{n+1}; \end{cases} \tag{6.9}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$, where λ, s are constants with $\lambda > 0$. Then there exists a unique reciprocal mapping $R : X^{n+1} \to Y$ such that

$$N\left(f(x) - R(x), r\right) \geq \begin{cases} N'\left(2\lambda, a, r\left|\varphi(2) - \varphi(n)\right|\right), \\ N'\left(2^{s+1}\lambda\beta(x), a, r\left|\varphi\left(2^{s+1}\right) - \varphi\left(n^{s+1}\right)\right|\right), \\ N'\left(2^{(n+1)s+1}\lambda\beta(x), r\left|\varphi\left(2^{(n+1)s+1}\right) - \varphi\left(n^{(n+1)s+1}\right)\right|\right), \end{cases} \tag{6.10}$$

for all $x, a \in X$ and all $r > 0$.

Proof: Setting $\alpha\left(x, y_1, y_2, ..., y_n\right) = \begin{cases} \lambda, \\ \lambda\left\{\|x\|^s + \sum_{i=1}^{n} \|y_i\|^s\right\}, \\ \lambda\left(\|x\|^s \prod_{i=1}^{n} \|y_i\|^s + \|x_i\|^{(n+1)s} + \sum_{i=1}^{n} \|y_i\|^{(n+1)s}\right) \end{cases}$

for all $x, y_1, y_2, ..., y_2 \in X$. Now,

$$N'\left(v_i^k \alpha\left(v_i^k x, v_i^k y_1, v_i^k y_2, ..., v_i^k y_n\right), a, r\right) = \begin{cases} N'\left(v_i^k \lambda, a, r\right), \\ N'\left(v_i^k \lambda\left\{\| v_i^k x\|^s + \sum_{i=1}^{n} \| v_i^k y_i\|^s\right\}, a, r\right), \\ N'\left(v_i^k \lambda\left(\| v_i^k x\|^s \prod_{i=1}^{n} \| v_i^k y_i\|^s + \| v_i^k x_i\|^{(n+1)s} + \sum_{i=1}^{n} \| v_i^k y_i\|^{(n+1)s}\right), a, r\right) \end{cases}$$

$$= \begin{cases} \to 1 \, as \;\; k \to \infty, \\ \to 1 \, as \;\; k \to \infty, \\ \to 1 \, as \;\; k \to \infty, \end{cases}$$

for all $x, y_1, y_2, ..., y_2, a \in X$ and all $r > 0$ .Thus, (4.1) is holds. But we have $\beta(x) = \alpha\left(\frac{nx}{2}, 0, ..., 0, 0\right)$, has the property $\beta(x) = L \cdot v_i \, \beta\left(v_i x\right)$ for all $x \in X$. Hence

$$\beta\left(x\right) = \alpha\left(\frac{nx}{2}, 0, ..., 0, 0\right) = \begin{cases} \lambda, \\ \lambda\left\{\left\|\frac{nx}{2}\right\|^s + 0 + ... + 0\right\}, \\ \lambda\left(0 + \left\|\frac{nx}{2}\right\|^{(n+1)s} + 0 + ... + 0\right) \end{cases}$$

Now, $$N'\left(v_i \beta\left(v_i x\right), a, r\right) = \begin{cases} N'\left(\lambda v_i, a, r\right), \\ N'\left(\lambda v_i^{s+1}\left(\frac{n}{2}\right)^s \|x\|^s, a, r\right), \\ N'\left(\lambda v_i^{(n+1)s+1}\left(\frac{n}{2}\right)^{(n+1)s} \|x\|^{(n+1)s}, a, r\right). \end{cases} = \begin{cases} N'\left(v_i \beta\left(x\right), a, r\right), \\ N'\left(v_i^{s+1} \beta\left(x\right), a, r\right), \\ N'\left(v_i^{(n+1)s+1} \beta\left(x\right), a, r\right). \end{cases}$$

Hence the inequality (6.3) holds either, $L = \left(2/n\right)$ if $i = 0$ and $L = \left(n/2\right)$ if $i = 1$, $L = \left(2/n\right)^{s+1}$ for $s < -1$ if $i = 0$ and $L = \left(n/2\right)^{s+1}$ for $s > -1$ if $i = 1$, $L = \left(2/n\right)^{(n+1)s+1}$ for $s < -\frac{1}{n+1}$ if $i = 0$ and $L = \left(n/2\right)^{(n+1)s+1}$ for $s > -\frac{1}{n+1}$ if $i = 1$. From (6.4), we arrive (6.10). Hence the proof is complete.

7. APPLICATIONS OF THE FUNCTIONAL EQUATIONS (1.1) AND (1.2)

Consider the additive functional equation (1.1), that is

$$f(x) = \sum_{l=1}^{n}\left(\frac{f\left(x + l y_l\right) + f\left(x - l y_l\right)}{2n}\right).$$

This functional equation can be used to find the n-consecutive terms of an arithmetic progression. Since $f(x) = x$ is the solution of the functional equation, the above equation is written as follows

$$x=\sum_{l=1}^{n}\left(\frac{(x+ly_l)+(x-ly_l)}{2n}\right).$$

Now, let us take the variables as consecutive terms, we note that the middle term of any n-consecutive terms of an arithmetic progression is always the arithmetic mean of the other n terms.

Any n consecutive terms of an arithmetic progression differ by the common difference, d . So any n consecutive terms of an arithmetic progression can be written as

$$b-nd,...,b-2d,b-d,b,b+d,b+2d,...,b+nd.$$

The middle term b can be represented by

$$b=\frac{(b-d)+(b+d)+(b-2d)+(b+2d)+...+(b-nd)+(b+nd)}{2n}.$$

i.e., b is the arithmetic mean of

$$(b-d)+(b+d)+(b-2d)+(b+2d)+...+(b-nd)+(b+nd).$$

Consider the reciprocal functional equation (1.2), that is

$$f\left(\frac{2x}{n}\right)=\sum_{l=1}^{n}\left(\frac{f(x+ly_l)f(x-ly_l)}{f(x+ly_l)+f(x-ly_l)}\right).$$

This functional equation can be used to find the n-consecutive terms of a harmonic progression. Since $f(x)=\frac{1}{x}$ is the solution of the functional equation, the above equation is written as follows

$$\frac{n}{2x}=\sum_{l=1}^{n}\left(\frac{\frac{1}{x+ly_l}\frac{1}{x-ly_l}}{\frac{1}{x+ly_l}+\frac{1}{x-ly_l}}\right).$$

Now, let us take the variables as n-consecutive terms, we note that half of the middle term of any n consecutive terms of a harmonic progression is always the division of product and sum of the other two terms.

Any n-consecutive terms of a harmonic progression differ by the common difference, d . So any n- consecutive terms of a harmonic progression can be written as

$$\frac{1}{b-nd},\cdots,\frac{1}{b-2d},\frac{1}{b-d},\frac{1}{b},\frac{1}{b+d},\frac{1}{b+2d},\cdots,\frac{1}{b+nd}.$$

The half of the middle term $\frac{1}{b}$ can be represented by

$$\frac{n}{2b}=\sum_{l=1}^{n}\left(\frac{\frac{1}{b+ld_l}\frac{1}{b-ld_l}}{\frac{1}{b+ld_l}+\frac{1}{b-ld_l}}\right).$$

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