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Group Theoretic Similarity Solution of Second Order Nonlinear Diffusion Equation in an Isotropic Medium

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ABSTRACT

In this paper the author discusses about the derivation of one-dimensional non-linear partial differential equation, which is associated with diffusion process. The infinitesimal transformations in searching for all possible groups of transformations to this partial differential equation are developed. The procedure is based on Lie's theory of infinitesimal continuous group of transformations. This group theory has been applied extensively in recent times by investigators in the field of similarity analysis. By this method, this partial differential equation is reduced into the Abel's equation of the second kind which is solved by the analytical method. The similarity solution is derived whenever the diffusivity coefficient varies linearly.

Keywords : Diffusion Equations, Similarity Solutions, Infinitesimal Transformations

I. INTRODUCTION

Nonlinear Diffusion Equations, an important class of parabolic equations, come from a variety of diffusion phenomena which appear widely in nature. They are suggested as mathematical models of physical problems in many fields, such as filtration, phase saturation, biochemistry and dynamics of biological groups. In many cases, the equations possess degeneracy or singularity. The appearance of degeneracy or singularity makes the study more in valued and challenging. Many new ideas and methods have been developed to overcome the special difficulties caused by the degeneracy and singularity, which enrich the theory of partial differential equations.

As can be seen, the second order diffusion equation, is very important due to its broad range of applications, and has hence been studied extensively using different analytical and numerical methods. Similarity solutions have been found by Michal [1], as well as solutions that illustrate the behavior of the waiting time phenomenon and solutions that have been constructed using singular perturbation theory. Numerical approximations include finite difference moving mesh methods, fixed grid methods with multi-grid solvers [2], and the adaptive mesh finite element method used for solving time-dependent partial differential equations. (Hansen [3]).

It was noted that the usual types of transformation for simplifying partial differential equations of physical problems were usually of a rather special class. This led to the question of possibly generalizing the class of transformations methods. A second area of investigation centered on broadening the definition of similarity to include other transformations which change a given problem into a simpler problem in some sense or other, instead of the usual interpretation of similarity which implies simply a reduction in the number of variables of a given problem. It has been known for some time that [1-3] the theory of continuous transformation groups gives promise of providing a very general method of analysis. The method is by no means new. In fact, the basic ideas date back to the last century and are found in the work of the mathematician Sophus Lie. (Cohen

[4]) The theory of continuous groups was first applied to the solution of partial differential equations by Birkhoff [5] who used a one-parameter group. Morgan [6] proved a theorem which established the condition under which the number of variables can be reduced by one. Morgan's theorem was later extended by Michal [1] to similarity transformations which reduce the number of independent variables by more than one. One of the difficulties of the approach by Birkhoff [5] and Morgan [6] is that it is based on particular given groups of transformations. To begin with, a group has to be arbitrarily assumed and Morgan's theorem can then be used to establish whether or not the differential equation transforms conformally under this particular group. If it does, similarity solutions will then exist and the similarity variables can then be taken as the functionally independent invariants of this group. The question which remains is whether or not there are other groups under which similarity solutions exist.

In this paper, the idea of Lie's infinitesimal transformation groups is applied to develop a method for searching possible groups of infinitesimal transformations, instead of arbitrarily assuming one at the outset. The method is then applied to a broader class of similarity analyses: namely, the similarity between partial and ordinary differential equations, boundary and initial value problems and nonlinear and linear differential equations.

II. MATHEMATICAL FORMULATION

Let us consider the flux of diffusing particles in one-dimension (x direction) illustrated in Figure 1. The particles can be atoms, molecules, or ions. Fick's first law in an isotropic medium can be written as

$$J_x = -D\frac{\partial u}{\partial x} \qquad (1)$$

Here J_x is the flux of particles (diffusion flux) and u their number density (concentration). The negative sign in Equation (1) indicates opposite directions of the diffusion flux and concentration gradient. Diffusion is the process which leads to an equalization of concentration. The factor of proportionality, D, is denoted as the diffusion coefficient or the diffusivity of the species considered. The diffusion flux can be expressed in number of particles or moles travelling a unit area per unit time and the concentration in number of particles per unit volume, therefore, the diffusion coefficient or diffusivity D has the dimension of $lenght^2 time^{-1}$.

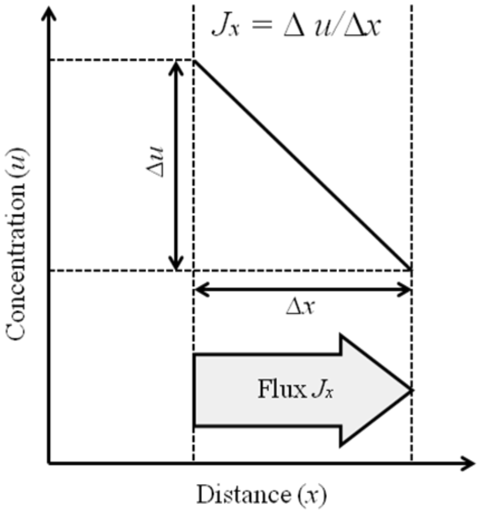

Figure 1

Fick's law can be easily generalized to three dimensions as

$$J = -D\nabla u \qquad (2)$$

The vector of the diffusion flux J is directed opposite in direction to the concentration gradient vector ∇u. The operator ∇ acts on the scalar concentration field $u(x, y, z, t)$ and produces the concentration gradient field ∇u. The concentration gradient vector always points in that direction for which the concentration field undergoes the most rapid increase and its magnitude equals the maximum rate of increase of concentration at the point. For an isotropic medium the diffusion flux is antiparallel to the concentration gradient. Equations (1) and (2) represent the simplest form of Fick's first law.

Usually, in diffusion processes the number of diffusing particles is conserved. This implies that the diffusing species neither undergo reactions nor exchange with internal sources or sinks. Sources and sinks are important for intrinsic point defects. For a diffusing species which obeys conservation law an equation of continuity can be formulated. To this end, let us choose an arbitrary point P located at (x, y, z) and a test volume of size Δx , Δy and Δz (Figure 2).

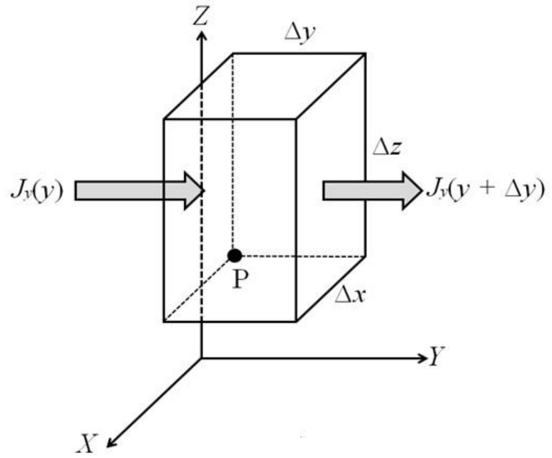

Figure 2

The diffusion flux J and its components J_x , J_y and J_z vary across the test volume. If the sum of the fluxes leaving and entering the test volume does not balance, a net accumulation (or loss) must occur. This material balance can be expressed as

inflow – outflow = accumulation (0r loss) rate

The flux components can be substituted into this equation to yield

$$[J_x(P) - J_x(P+\nabla x)]\nabla y \nabla z + [J_y(P) - J_y(P+\nabla y)]\nabla x \nabla z + [J_z(P) - J_z(P+\nabla z)]\nabla x \nabla y = accumulation\ (or loss)\ rate$$

Using Taylor expansions of the flux components up to their linear terms, the expressions in square brackets can be replaced by $\Delta x \frac{\partial J_x}{\partial x}$, $\Delta y \frac{\partial J_y}{\partial y}$ and $\Delta z \frac{\partial J_z}{\partial z}$ respectively.

This yields,

$$-\left[\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}\right]\nabla x \nabla y \nabla z = \frac{\partial u}{\partial t}\nabla x \nabla y \nabla z \qquad (3)$$

where the accumulation (or loss) rate in the test volume is expressed in terms of the partial time derivative of the concentration. For infinitesimal size of the test volume Eq. (3) can be written in compact form by introducing the vector operator divergence ∇ , which acts on the vector of the diffusion flux:

$$-\nabla \cdot J = \frac{\partial u}{\partial t} \qquad (4)$$

where u is concentration and J is the flux.

Equation (4) is denoted as the continuity equation.

Fick's first law Eq. (2) and the equation of continuity (4) can be combined to give an equation

$$\frac{\partial u}{\partial t} = \nabla \cdot (D \nabla u) \qquad (5)$$

which is called Fick's second law or the diffusion equation.

From a mathematical viewpoint Eq. (5) is a second-order partial differential equation. It is nonlinear if D depends on concentration, which is, for example, the case when diffusion occurs in a chemical composition gradient. The composition dependent diffusivity is usually denoted as the inter-diffusion coefficient. For arbitrary composition dependence $D(u)$, Eq. (5) usually cannot be solved analytically. Eq. (5) can be rewritten for one dimension (x direction) as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) \qquad (6)$$

which is the one dimensional nonlinear diffusion equation of concentration in an isotropic medium.

III. SIMILARITY SOLUTION

Now we discuss a specific case for the above equation (6), i.e. let us take $D(u) = u$, on which an infinitesimal transformation is to be made on the dependent and independent variables and derivatives of the dependent variable with respect to the independent variable.

$$\therefore \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right)$$

$$\therefore \frac{\partial u}{\partial t} = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2} \qquad (7)$$

Let the one parameter group of infinitesimal transformations be

$$G = \begin{cases} \bar{x} = x + \in X \\ \bar{t} = t + \in T \\ \bar{u} = u + \in U \end{cases} \qquad (8)$$

where generators X , T and U are the functions of x , t and u .

Invariance of equation (1) under (8) gives,

$$\therefore \frac{\partial \bar{u}}{\partial \bar{t}} = \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} \qquad (9)$$

Applying transformations (8) into (9), we get the group of infinitesimal transformation explicitly is

$$\begin{cases} T = at + b \\ X = \frac{ax}{2} + k \\ U = 0 \end{cases} \quad (10)$$

where a, b and k are constants.

Let us tale $a = 2$ and $b = k = 0$ to make the result (10) more simpler, then we have,

$$\begin{cases} T = 2t \\ X = x \\ U = 0 \end{cases} \quad (11)$$

Then the characteristic equations are,

$$\frac{dx}{X} = \frac{dt}{T} = \frac{du}{U}$$

$$\therefore \frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0} \quad (12)$$

From the first equation we get the similarity variable,

$$\eta = \frac{x}{\sqrt{t}} \quad (13)$$

Now putting $u = f(\eta)$, where $\eta = \frac{x}{\sqrt{t}}$

nto an equation (7) we get,

$$f(\eta) f''(\eta) + \left[f'(\eta)\right]^2 + \frac{\eta}{2} f'(\eta) = 0 \quad (14)$$

It is the non-linear second order ordinary differential equation for the phenomenon under consideration.

Consider $f(\eta) = \eta^2 w(z)$ [7]

Where $z = \log_e \eta$

$\Theta\, w'(z) = p$

Then the above differential equation reduces to

$$wpp'(w) + p^2 + \left(\frac{1}{2} + 7w\right)p + w(1 + 6w) = 0$$

which is the Abel's of second kind and its solution is given by[7];

$$\int w\left(w^2 + w + c\right)dw = \int \frac{(1+14\eta)^2}{4\eta^2(1+6\eta)} d\eta \quad (15)$$

Where c is constant and $w(\eta) = \frac{1}{\eta f(\eta)}$ (16)

From the results (15) and (16), we get the similarity solution of the equation (14).

IV. CONCLUSION

Here the author has discussed a specific problem of diffusion under the certain assumptions in an isotropic medium and obtained the similarity solution using the technique of groups of infinitesimal transformations. The similarity solution is derived whenever the diffusivity coefficient varies linearly.

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