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Existence and Uniqueness of solutions of Fractional Order Hybrid Differential Equations with Nonlocal Conditions

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ABSTRACT

In this article we derive sufficient conditions for existence and uniqueness of mild solution of the hybrid fractional order differential equation of the form

$$^cD^\alpha[x(t) - f(t, x(t))] = g(t, x(t)), \qquad t \in [t_0, T] \;\; a.e.$$

$$x(t_0) = h(x)$$

On a partially ordered Banach space X. The results are obtained using Dhange's fixed point theorem. Example is added to show efficacy of the method.

Keywords: Fractional Differential Equations, Hybrid Systems, Fixed Point Theorem

1 Introduction

From the past decades fractional differential equations is one of the renowned subject due to its various applications in all sciences [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. This is due to their nonlocal property means, the successive state is depend on all its previous states [15] and this nonlocal property helped to remodel their systems in terms of fractional order systems [16]. Existence and uniqueness of solution of fractional order differential equation with classical condition was initiated by Delbosco and Rodino [17] and subsequently extended results by several researchers [15, 18, 19, 20, 21, 22, 23, 24, 25].
The systems having both continuous and discrete behavior are called hybrid systems. Due to its dual nature many researchers took interest to study these hybrid systems. Existence and uniqueness of solution of hybrid first order systems using fixed point theorem was extensively studied by Krasnoselskii, Burton and Dhage [31, 33, 34, 35].
The study of existence and uniqueness of solution of Caputo fractional hybrid differential equations was initiated by Herzallah and Beleanu [37].Recently, existence and uniqueness of fractional order systems with classical condition was studied by Somjaiwang and Ngiamsunthorn [36].
In this article we are modifying the work of Somjaiwang and Ngiamsunthorn and study existence and uniqueness of mild solution of fractional hybrid system with nonlocal conditions

$$\begin{aligned} ^cD^\alpha[x(t) - f(t, x(t))] &= g(t, x(t)) \quad t \in [t_0, T] \; a.e \\ x(0) &= h(x) \end{aligned} \qquad (1.1)$$

on the ordered space X because nonlocal conditions can more relevant then classical conditions to describe some physical phenomena.

2 Priliminaries

Some basic definitions and properties of fractional calculus and fractional differential equations used in this article, are as follows:

Definition 2.1. *The Riemann-Liouville fractional integral operator of $\alpha > 0$, of function $f \in L_1(\mathbb{R}_+)$ is defined as*

$$I^\alpha_{t_0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds,$$

provided the integral on right side exist. Where $\Gamma(\cdot)$ *is gamma function [29].*

Definition 2.2. *The Riemann-Liouville fractional derivative of order* $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, *is defined as*

$$D^{\alpha}_{t_0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(t-s)^{n-\alpha-1}f(s)ds,$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$ *[29].*

This derivative has singularity at zero and also requires special initial condition which lacks physical interpretation. To overcome this difficulty, Caputo [26] interchanged the role of operators and defined the fractional derivatives as follows:

Definition 2.3. *The Caputo fractional derivative of order* $\alpha > 0$, $n-1 < \alpha < n$, $n \in \mathbb{N}$, *is defined as*

$${}^{c}D^{\alpha}_{t_0+}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-s)^{n-\alpha-1}\frac{d^n f(s)}{ds^n}ds,$$

where the function $f(t)$ *has absolutely continuous derivatives up to order* $(n-1)$ *[30].*

Fractional integral and differential operator satisfy following properties which is mentioned in Kilbas *et. al.* [27] and Samko *et. al.* [28].

Theorem 2.1. *For* $\alpha, \beta > 0$ *and* f *having absolutely continuous derivatives up to suitable order, then*
$(1) I^{\alpha}_{t_0+}I^{\beta}_{t_0+}f(t) = I^{\alpha+\beta}_{t_0+}f(t)$
$(2) I^{\alpha}_{t_0+}I^{\beta}_{t_0+}f(t) = I^{\beta}_{t_0+}I^{\alpha}_{t_0+}f(t)$
$(3) I^{\alpha}_{t_0+}(f(t)+g(t)) = I^{\alpha}_{t_0+}f(t) + I^{\alpha}_{t_0+}g(t)$
$(4) I^{\alpha}_{t_0+}\,{}^{c}D^{\alpha}_{t_0+}f(t) = f(t) - f(0), \quad 0 < \alpha < 1$
$(5) {}^{c}D^{\alpha}_{t_0+}I^{\alpha}_{t_0+}f(t) = f(t)$
$(6) {}^{c}D^{\alpha}_{t_0+}f(t) = I^{1-\alpha}_{t_0+}f'(t), \quad 0 < \alpha < 1$
$(7) {}^{c}D^{\alpha}_{t_0+}\,{}^{c}D^{\beta}_{t_0+}f(t) \neq {}^{c}D^{\alpha+\beta}_{t_0+}f(t)$
$(8) {}^{c}D^{\alpha}_{t_0+}\,{}^{c}D^{\beta}_{t_0+}f(t) \neq {}^{c}D^{\beta}_{t_0+}\,{}^{c}D^{\alpha}_{t_0+}f(t)$

For convenience, ${}^{c}D^{\alpha}_{0+}$ is taken as ${}^{c}D^{\alpha}$.

2.1 Notations

(N1) X = Banach space equipped with partial order.

(N2) $\mathbb{R}_+ = [0, \infty)$

(N3) $C([0,T_0], X) = \{x : [0,T_0] \to X / x \text{ is continuous}\}$ with norm $||x|| = sup_t||x(t)||$

Definition 2.4. *([31]) An operator* $T : X \to X$ *is called nondecreasing if the order relation preserved under* T*, this means, for each* $x, y \in X$ *such that* $x \le y$ *implies* $Tx \le Ty$.

Definition 2.5. *([31]) The order relation* $\le$ *and a metric* d *are compatible on nonempty set* X *then convergence of subsequence* $\{x_{n_k}\}$ *implies the convergence of* $\{x_n\}$ *for any monotone sequence* $\{x_n\}$ *in* X.

Definition 2.6. *([31]) An upper semi-continuous and nondecreasing function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is* $\mathcal{D}$*-function if* $\psi(0) = 0$

Definition 2.7. *([32]) An operator* $T : X \to X$ *is called partially continuous at* $a \in X$ *if for any* $\epsilon > 0$*, there exist* $\delta > 0$ *such that* $||Tx - Ta|| < \epsilon$ *for all* x *comparable to* a *in* X *with* $||x - a|| < \delta$ *and* T *is continuous on* X *if* T *is partially continuous at every* $a \in X$*. In particular, if* T *is partially continuous on* X*, then it is continuous on every chain* C *in* X*. An operator* T *is called partially bounded if* $T(C)$ *is bounded for every chain* C *in* X*. An operator* T *is said to be uniformly partially bounded if all the chains* $T(C)$ *in* X *are bounded.*

Definition 2.8. *([32]) An operator* T *on* X *is called partially compact if for any chain* C *in* X*, the set* $T(C)$ *are relatively compact subset of* X*. An operator* T *is said to be partially totally bounded if for any totally ordered and bounded subset* C *of* X*, the set* $T(C)$ *is a relatively compact subset of* X*. If* T *is partially continuous and partially totally bounded then* T *is partially completely continuous operator on* X*.*

Definition 2.9. *([32]) A mapping* $T : X \to X$ *is partially nonlinear* $\mathcal{D}$*-Lipschitz if there is a* $\mathcal{D}$*-function* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $||Tx - Ty|| \le \psi||x - y||$ *for all compatible points* $x, y \in X$*. If* $\psi(r) < r$ *then* T *is called* $\mathcal{D}$*-contraction.*

Theorem 2.2. *([32]) Let order and norm are compatible on a partially order Banach space. Let* P *and*

Q are nondecreasing operators on X such that:
(a) P is partially bounded nonlinear $\mathcal{D}$-contraction.
(b) Q is partially continuous and partially compact.
(c) There exist an element $x_0 \in X$ such that $x_0 \leq Px_0 + Qx_0$. Then there exist a solution x^* in X of operator equation $Px + Qx = x$. In addition, the sequence $\{x_n\}$ of successive iteration $x_{n+1} = Px_n + Qx_n$ converges monotonically to x^*.

Definition 2.10. *Function $x(t) \in X$ is called mild solution of fractional order hybrid system 1.1 if $x(t)$ satisfies,*

$$\begin{aligned} x(t) &= h(x) - f(t_0, h(x)) + f(t, x(t)) \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, x(s)) ds \quad (2.1) \end{aligned}$$

for all value of $t \in [t_0, T]$.

3 Assumptions

(A1) The functions $f : [t_0, T] \times X \to X$ and $g : [t_0, T] \times X \to X$ are nondecreasing and continuous.

(A2) The function $h : X \to X$ is continuous and nondecreasing in x.

(A3) Function f is $\mathcal{D}$-contraction with ϕ such that $\phi(r) < r$.

(A4) There exist a constant M_f such that $|f(t,x)| \leq M_f$ for all $t \in [t_0, T]$ and $x \in X$.

(A5) For each $x, y \in X$ with $x \geq y$, $h(x) - h(y) \geq f(t_0, h(x)) - f(t_0, h(y))$.

(B1) There exist a positive constants M_g and M_h such that $|g(t,x)| \leq M_g$ and $|h(x)| \leq M_h$.

(B2) There exist a function $u \in X$ such that u is lower solution of the equation 1.1.

4 Main Result

In this section we discuss the existence and uniqueness of mild solution of fractional order hybrid system with nonlocal condition 1.1.

Theorem 4.1. *If assumptions (A1)-(A4) and (B1)-(B3) are satisfied then fractional order hybrid system 1.1 has unique mild solution in partially order Banach space X.*

Proof. We use theorem 2.2 to prove existence and uniqueness of mild solution of 1.1. Defining $Px(t) = f(t,x)$ and $Q = h(x) - f(t_0, h(x)) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, x(s)) ds$ the equation 2.1 becomes $x = Px + Qx$. By theorem 2.2, this operator equation has unique solution if conditions in hypotheses of the theorem 2.2 is satisfied.

Step-1
By assuming the conditions (A1),(A2) and (B1), we can easily prove that the operators P and Q are nondecreasing. Also by assuming (A1)-(A4), we can prove that P is partially bounded nonlinear $\mathcal{D}$-contraction on X.

Step-2
In this section we prove that Q is partially continuous and partially compact on X. Let, $\{x_n\}$ be a sequence in a chain C in X such that $x_n \to x$. Then,

$$\begin{aligned} ||Qx_n(t) - Qx(t)|| &\leq ||hx_n(t) - hx(t)|| \\ &+ ||f(t_0, x_n(t)) - f(t_0, x(t))|| \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \\ &|| \quad g(s, x_n(s)) - g(s, x(s))||ds. \end{aligned}$$

Assuming continuity of f, h and g and dominated convergence theorem, $||Qx_n(t) - Qx(t)|| \to 0$ as $n \to \infty$ for all $t \in [t_0, T]$.
Moreover the sequence $\{Qx_n\}$ is equicontinuous on X as let, $t_1, t_2 \in [t_0, T]$ with $t_1 < t_2$ then,

$$\begin{aligned} |Qx_n(t_2) &- Qx_n(t_1)| \leq |h(x_n(t_2)) - h(x_n(t_1))| \\ &+ |f(t_0, h(x_n(t_2))) - f(t_0, h(x_n(t_1)))| \\ &+ \Bigg| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} [(t_2 - s)^{\alpha-1} \\ &- (t_1 - s)^{\alpha-1}] g(s, x_n(s)) ds \Bigg| \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |g(s, x_n(s))| ds. \end{aligned}$$

Assuming conditions (A1),(A2) and (B1), $|Qx_n(t_2) - Qx_n(t_1)| \to 0$ as $t_2 - t_1 \to 0$. Therefore, the sequence

$\{Qx_n\}$ is equicontinuous. Thus $Qx_n \to Qx$ is uniformly on chain C and hence, Q is partially continuous on X.

Step -3

In this step we will show that Q is partially compact. Let $x \in C$ where C is chain in X. Then,

$$\begin{aligned} \|Qx(t)\| &\le \|h(x)\| + \|f(t_0, h(x))\| \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|g(s, x(s))\| ds \\ &\le M_h + M_f + \frac{M_g}{\alpha\Gamma(\alpha)} (T - t_0)^{\alpha} = K \end{aligned}$$

for all $t \in [t_0, T]$. Therefore $\|Qx(t)\| \le K$ and hence $Q(C)$ is uniformly bounded.

Since,

$$\begin{aligned} |Qx(t_2) &- Qx(t_1)| \le |h(x(t_2)) - h(x(t_1))| \\ &+ |f(t_0, h(x(t_2))) - f(t_0, h(x(t_1)))| \\ &+ \left| \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \right. \\ &\left. g(s, x(s)) ds \right| + \frac{1}{\Gamma(\alpha)} \\ &\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} |g(s, x(s))| ds \end{aligned}$$

and $|Qx(t_2) - Qx(t_1)| \to 0$ as $t_2 \to t_1$ uniformly for all $x \in C$. Therefore $Q(C)$ is relatively compact. Hence, Q is partially compact.

Step-4

By hypotheses (B2), the fractional hybrid system has lower solution u defined on $[t_0, T]$. That is

$$\begin{aligned} {}^cD^{\alpha} \left[u(t) - f(t, u(t)) \right] &= g(t, u(t)) \\ u(0) &\le u(x). \end{aligned}$$

Applying integral operator both side we get,

$$\begin{aligned} u(t) &\le h(u) - f(t_0, h(u)) + f(t, u(t)) \\ &+ \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s, u(s)) ds \end{aligned}$$

for all $t \in [t_0, T]$. Therefore $u \le Pu + Qu$.

Thus, from these steps we can conclude that the operators P and Q satisfy all the conditions of the hypotheses of the theorem 2.2 and hence the equation 1.1 has unique mild solution. □

Example 4.1.1. *Consider the hybrid partial differential equation on partially ordered Banach space*

$$\begin{aligned} {}^cD_t^{\alpha}[z(t,y) &- tz^2(t,y)] = \frac{\partial^2 z(t,y)}{\partial y^2} + \int_0^t z^2(s,y) ds \\ z(t,0) &= z(t,\pi) = 0 \\ z(0,y) &= \int_0^1 h(s) sin(1 + z(t,y)) ds \qquad (4.1) \end{aligned}$$

$t \in [0,T]$ *a.e.then,* (4.1) *can be represented in terms of* (1.1) *by taking* $x(t) = z(t,\cdot)$, $f(t,x) = tz^2(t,y)$, $g(t,x) = \frac{\partial^2 z(t,y)}{\partial y^2} + \int_0^t z^2(s,y)ds$, $h(x) = \int_0^1 h(s)sin(1 + z(t,y))$ *and* $x(0) = h(x)$ *Clearly,* f, g *and* h *satisfied assumptions of the theorem therefore* (4.1) *has unique mild solution.*

5 Conclusion

In this article authors studied existence and uniqueness of fractional hybrid system with nonlocal conditions using Dhange's fixed point theorem which will be more practicable than classical conditions.

References

[1] M. Renardy, W. J. Hrusa, J. A. Nohel, Mathematical problems in viscoelasticity, Longman Scientific and technical, Newyork, (1987).

[2] J. H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering, 167(1998), 57-68.

[3] A. M. A. El-Sayed, Fractional order wave equation, International Journal of Theoretical Physics, 35(1996),311-322.

[4] V. Gafiychuk, B. Datsan, V. Meleshko, Mathematical modeling of time fractional reaction-diffusion system, Journal of Computational and Applied Mathematics, 220(2008), 215-225.

[5] R. Metzler, J. Klafter, The restaurant at the end of random walk, the recent developments in description of anomalous transport by fractional dynamics, Journal of Physics A: A Mathematical and General, 37(2004), 161-208

[6] J. H. He,Some applications of nonlinear fractional differential equations and their approximations, Bulletin of Science and Technology, 15(1999), 8690.

[7] K. Sayevand, M. Fardi, E. Moradi, F. Hemati Boroujeni, Convergence analysis of homotopy perturbation method for Volterra integro-differential equations of fractional order, Alexandria Engineering Journal (2013) 52, 807-812.

[8] J. Prakash, M. Kothandapani, V. Bharathi, Numerical approximations of nonlinear fractional differential difference equations by using modified He-Laplace method, Alexandria Engineering Journal (2016) 55, 645-651.

[9] Mohammad Tamsir, Vineet K. Srivastava, Revisiting the approximate analytical solution of fractional order gas dynamics equation, Alexandria Engineering Journal (2016) 55, 867-874.

[10] Olaniyi Samuel Iyiola, Gbenga Olay, The fractional Rosenau-Hyman model and its approximate solutioninka Ojo, Okpala Mmaduabuchi, Alexandria Engineering Journal (2016) 55, 1655-1659.

[11] Jyotindra C. Prajapati, Krunal B. Kachhia, Shiv Prasad Kosta, Fractional calculus approach to study temperature distribution within a spinning satellite, Alexandria Engineering Journal (2016) 55, 2345-2350.

[12] Harendra Singh, A new numerical algorithm for fractional model of Bloch equation in nuclear magnetic resonance, Alexandria Engineering Journal (2016) 55, 2863-2869.

[13] Muhammad Saqib, Farhad Ali, Ilyas Khan, Nadeem Ahmad Sheikh, Syed Aftab Alam Jan, Samiulhaq, Exact solutions for free convection flow of generalized Jeffrey fluid: A Caputo-Fabrizio fractional model, Alexandria Engineering Journal (2017) (in press).

[14] Nauman Raza, M. Abdullah, Asma Rashid Butt, Aziz Ullah Awan, Ehsan Ul Haque, Flow of a second grade fluid with fractional derivatives due to a quadratic time dependent shear stress, Alexandria Engineering Journal (2017) (in press).

[15] Y. Cheng, G. Guozhu, On the solution of nonlinear fractional order differential equations, Nonlinear Analysis: Theory, Methods and Applications, 310(2005), 26-29.

[16] B. Bonilla, M. Rivero, L. Rodriguez-Germa, J. J. Trujillo, Fractional differential equations as alternative models to nonlinear differential equations, Applied Mathematics and Computation 187 (2007) 7988.

[17] D. Delbosco, L. Rodino, Existence and uniqueness for nonlinear fractional differential equations, Journal of Mathematical Analysis and Applications, 204(1996), 609-625. ry, Methods and Applications, 310(2005), 26-29.

[18] M. M. El-Borai, Semigroups and some nonlinear fractional differential equations, Applied Mathematics and Computations, 149(2004), 823-831.

[19] L. Byszewski, Theorems about the existence and uniqueness of solutions of semi linear evolution non-local Cauchy problem, Journal of Mathematical Analysis and Applications, 162(1991), 494-505.

[20] K. Balachandran, J. J. Trujillo, The non-local Cauchy problem for nonlinear fractional integro-differential equations in Banach spaces, Nonlinear Analysis, 72(2010), 4587-4593.

[21] M. Benshohra, J. Henderson, S. K. Ntouyas, A. Quahab, Existence results for fractional order functional differential equations with infinite delay, Journal of Mathematical Analysis and Applications,338(2008),1340-1350.

[22] L. Mahto, S. Abbas, Existence and uniqueness of Caputo fractional differential equations, AIP Conf. Proc., 1479(2012), 896-899.

[23] A. Neamaty, M. Yadollahzadeh, R. Darzi, On fractional differential equation with complex order, Prog. Frac. Differ. Appl. 1(2015),223-227.

[24] R. Ibrahim, A. Kilicman, F. Damag, Existence and uniqueness for a class of iterative fractional differential equations, 78(2015), 1-13.

[25] M. Matar, J. Trujillo, Existence of local solutions for differential equations with arbitary fractional order, Arab. J. Math. 5(2016), 215-224.

[26] Caputo M., Linear model of dissipation whose Q is almost frequency independent, Part II. Geophysical Journal of Royal Astronomical Society, 13(1967), 529-539.

[27] Kilbas A. A., Srivastava H. M., Trujillo J. J, Theory and Applications of Fractional Differential Equations, Elsevier Science, (2006).

[28] Samko S. G. Kilbas A. A, Marichev O. I, Fractional Integrals and Derivatives; Theory and Applications, Gorden and Breach, (1993).

[29] K. Balachandran, S. Kiruthika, J. J. Trujillo, Existence results for fractional impulsive integro-differential equations in Banach spaces, Commun Nonlinear Sci Numer Simulat, 16(2011), 1970-1977.

[30] K. Balachandran, S. Kiruthika, J. J. Trujillo, On fractional impulsive equations of sobolev type with nonlocal condition in Banach spaces, Computer and Mathematics with Applications, 62(2011),1157-1165.

[31] B. Dhage, S. Dhage, Hybrid fixed point theory ordered normed linear spaces and applications to fractional integral equations, Differ. Equ. Appl., 5(2013), 155-184.

[32] B. Dhage, Partially condensing mappings in partially order normed linear spaces and applications to functional integral equation, Tamkang J. Math., 45(2014), 397-426.

[33] M. A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, 1964.

[34] T. A. Burton, A fixed point theorem of Krasnoselskii, Appl. Math. Lett., 11 (1998), 8388.

[35] B. C. Dhage and V. Lakshmikantham, Basic results on hybrid differential equations, Nonlinear Analysis: Hybrid Systems, 4 (2010), 414424.

[36] D. Somjaiwang and P. S. Ngiamsunthorn, Existence and approximations of solutions to fractional order hybrid differential equations, Advances in Differential equations, 2016(2016) 1-13.

[37] M. Herzallah and D. Baleanu, On fractional order hybrid differential equations, Abstr. Appl. Anal., 2014(2014), Aricle ID-389386.