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Solution of Ordinary Differential Equation Using Green's Function & Sturm – Liouville Problem

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ABSTRACT

In this paper, we describe Green's function to determine the importance of this function, i.e. Boundary & Initial Value problem, Sturm-Liouville Problem. Along with the series representation of Green's Function.

Keywords : Boundary Value Problem, Green's Function, Sturm-Liouville Differential Equation.

I. INTRODUCTION

The series solution of differential equation yields an infinite series which often converges slowly. Thus it is difficult to obtain an insight into over-all behaviour of the solution. The Green's function approach would allow us to have an integral representation of the solution instead of an infinite series.

A Green's function G(x, s) of linear differential operator L = L(x) acting on distributions over a subset of the Euclidean space R^N at a point s, is any solution of LG(x, s) = δ(s-x)

Where δ is the Dirac delta function.

This Property of a Green's function can be exploited to solve differential equations of the form Lu(x) = f(x)

II. PRELIMINARIES

A. Sturm-Liouville Problem

Consider a linear second ordinary differential equation

$$A(x)\frac{d^2y}{dx^2} + B(x)\frac{dy}{dx} + C(x)\, y + \lambda\, D(x)\, y = 0 \qquad (1)$$

Where λ is a parameter to be determined by the boundary conditions. A(x) is positive continuous function, then by dividing every term A(x), equation (1) can be written as

$$\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)\, y + \lambda\, d(x)\, y = 0 \qquad (2)$$

Where $b(x) = \frac{B(x)}{A(x)}$, $c(x) = \frac{B(x)}{A(x)}$ and $d(x) = \frac{D(x)}{A(x)}$

Let us define integrating factor P(x) by

$$P(x) = \exp\{\int_a^x b(\varepsilon)d\varepsilon\}$$

Multiplying (2) by P(x), we have

$$P(x)\frac{d^2y}{dx^2} + P(x)b(x)\frac{dy}{dx} + P(x)c(x)y + \lambda\, P(x)d(x)y = 0 \quad (3)$$

Since $\frac{dP(x)}{dx} = \frac{d}{dx} e^{\int_a^x b(\zeta)d\zeta}$

$= e^{\int_a^x b(\zeta)d\zeta} \frac{d}{dx} [\int_a^x b(\zeta)d\,\zeta]$

$= P(x)\, b(x)$

So $\frac{d}{dx}[P(x)\frac{dy}{dx}] = P(x)\frac{d^2y}{dx^2} + \frac{dP(x)}{dx}\frac{dy}{dx}$

$= P(x)\frac{d^2y}{dx^2} + P(x)\, b(x)\, Z$

Thus equation (3) can be written as

$$\frac{d}{dx}[P(x)\frac{dy}{dx}] + q(x)\, y + \lambda\, r(x)\, y = 0 \qquad (4)$$

where q(x) =p(x) c(x) and r(x) =p(x) dx

Equation in form (4) is known as Sturm-Liouville equation. Satisfy the boundary conditions.

Regular Sturm-Liouville problem

In case p (a) ≠0 & p (b) ≠0, p(x), q(x), r(x) are continuous, the Sturm-Liouville equation (4) can be expressed as

$$L(y) = \lambda\, r(x)\, y \quad (5)$$

$$\text{Where } L = \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + q(x) \quad (6)$$

If the above equation is associated with the following boundary condition

$\alpha_1\, y(a) + \alpha_2\, y^1(a) = 0$

$$\beta_1\, y(a) + \beta_2\, y^1(a) = 0 \quad (7)$$

The equation (4) & the boundary condition (7) are called Regular Sturm-Liouville Problem (RSLP)

B. Green's Function

1) 1) The Concept of Green's Function

In the case of ordinary differential equation we can express the problem as

$L[y] = f$

Where L is a linear differential operator f(x) is known function & y(x) is desired solution. We will show that the solution y(x) is given by an integral involving that Green's function G(x, ε).

2) Green's Function For Ordinary Differential Equation

Here we consider non-homogeneous ordinary differential equation

$L[y] = f$

Where L is an ordinary linear differential operator that can be represented by Sturm-Liouville operator that is

$$L = \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + q(x)$$

And the Sturm-Liouville type is given by

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + q(x)\, y + \lambda\, r(x)\, y = -f(x) \quad (8)$$

Where λ is a parameter. Now consider the linear non homogeneous ordinary differential equation of the form

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + q(x)\, y = -f(x) \quad a<x<b$$

With the boundary condition

$\alpha_1\, y(a) + \alpha_2\, y^1(a) = 0$

$$\beta_1\, y(a) + \beta_2\, y^1(a) = 0 \quad (9)$$

where the constant are such that $\alpha_1 + \alpha_2 \neq 0$ & $\beta_1 + \beta_2 \neq 0$ if λ=0 then equation (8) are identical in the interval & r(x) are real & positive in that interval.

Now we are seeking to determine the Green's function G for the equation satisfies the following

$$\frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + q(x)\, y = -\delta\,(s - \zeta)$$

With the boundary condition

$\alpha_1\, G(a) + \alpha_2\, G^1(a, \zeta) = 0$

$\beta_1\, G(a) + \beta_2\, G^1(a, \zeta) = 0$

Now consider the region $a \leq x \leq \zeta$

Let $y_1(x)$ be a non trivial solution at x=a, i.e.

$\alpha_1 y_1(a) + \alpha_2\, y^1(a) = 0$

$$\text{then } \alpha_1\, y_1(a, \zeta) + \alpha_2\, G^1(a, \zeta) = 0 \quad (10)$$

The wronskianof y_1& G must vanish at x=a

$y_1(a)\, G^1(a, \zeta) + y^1(a)\, G^1(a, \zeta) = 0$

$$\text{so } G(x, \zeta) = u_1 y_1(x) \text{ for } a<x<\zeta \quad (11)$$

where u_1 is an arbitrary constant. Similarly if the non trivial solution $y_2(x)$ satisfies the homogeneous equation and the condition at x=b, then

$G(x, \zeta) = u_1 y_1(x)$ for $a<x<\zeta$

Now by integrating from ζ – ε to ζ + ε we obtain

$$P(x)\, \frac{dG(x,\zeta)}{dx}\left[\begin{matrix}\zeta + \varepsilon \\ \zeta - \varepsilon\end{matrix} + q(x)G(x,\zeta)dx\right] = -1 \quad (12)$$

Since G(x, ζ) and q(x) are continuous at x=ζ then we have

$$\frac{dG(\zeta^{+},\zeta)}{dx} = \frac{dG(\zeta,\zeta)}{dx} = \frac{1}{p(\zeta)} \quad (13)$$

The continuity condition of G & the Jump discontinuity of G^1 at x=ζ. From equations (10), (11) & (12)

$u_1\, y_1\,(\zeta) - u_2\, y_2 = 0$

$u_1\, y_1\,(\zeta) - u_2 y_2^1 = \frac{1}{p(\zeta)}$

we can solve (12) for u_1 and u_2 provided the wronskian y_1 and y_2 does not vanish at x=ζ or

$y_1\,(\zeta)\, y_2^1(\zeta)\, y_1^1(\zeta) \neq 0$

i.e., $w(\zeta) = \begin{vmatrix} y_1(\zeta) & y_1(\zeta) \\ y_1{}^1(\zeta) & y_2{}^1(\zeta) \end{vmatrix} \neq 0$

The system of equation (13) has the solution

$U_1 = \frac{-y_2{}^1(\zeta)}{p(\zeta)w(\zeta)}$

$U_2 = \frac{-y_1{}^1(\zeta)}{p(\zeta)w(\zeta)}$

Where w(ζ) is the wronskian of $y_1(x)$ and $y_2(x)$ at x=ζ

$G(x,\zeta) = \frac{y_1(x<)y_2(x>)}{p(\zeta)w(\zeta)}$

$$G(x,\zeta) = \frac{y_1(x)y_2(\zeta)}{p(\zeta)w(\zeta)}\ \frac{y_1(x<)y_2(x>)}{p(\zeta)w(\zeta)} \qquad (14)$$

Now from (14) the solution (8) can be

$y(x) = \int_a^b G(x,\zeta)f(\zeta)d\zeta$

so $y(x) = \int_a^x G(x,\zeta)f(\zeta)d\zeta + \int_x^b G(x,\zeta)f(\zeta)d\zeta$

Example.

Solve the boundary value problem y'' = x^2, y (0) = 0= y (1) using the boundary value Green's function

We first solve the homogeneous equation, y''=0. After two integrations, we have y(x) = Ax + B, for A and B constants to be determined.

We need one solution satisfying $y_1(0)$. Thus,

$0 = y_1(0) = B$

So, we can pick $y_1(x) = x$, since A is arbitrary.

The other solution has to satisfy $y_2(1) = 0$. So,

$0 = y_2(1) = A + B$

This can be solved for B= - A. Again A is arbitrary and we will choose A= - 1.

Thus, $y_2(x) = 1-x$

For this problem p(x) = 1. Thus for $y_1(x) = x$ and $y_2(x) = 1 - x$,

$P(x)W(x)=y_1(x)y_2'(x)-y_1'(x)y_2(x)= x(-1) - 1(1-x)= -1$

Note that p(x) w(x) is a constant as it should be.

Now we construct the Green's function. We have

$$G(x,\zeta) = \begin{cases} -\zeta(1-x), 0 \le \zeta \le x \\ -x(1-\zeta), x \le \zeta \le 1 \end{cases}$$

Notice, the symmetry between the two branches of the Green function. Also, the Green's function satisfies homogeneous boundary conditions: G (0, ζ) = 0, from the lower branch, and G (1, ζ) = 0 from the upper branch.

Finally, we insert the Green's function into the integral form of the solution and evaluate the integral

$y(x) = \int_0^1 G(x,\zeta)\, f(\zeta)\, d\zeta$

$= \int_0^1 G(x,\zeta)\, \zeta^2\, d\zeta$

$= \int_0^x -\zeta(1-x)\, \zeta^2\, d\zeta - \int_x^1 x(1-\zeta)\, \zeta^2\, d\zeta$

$= -(1-x)\int_0^x \zeta^3\, d\zeta - x\int_x^1 (\zeta^2 - \zeta^3)\, d\zeta$

$= -(1-x)\left[\frac{\zeta^4}{4}\right]_0^x - x\left[\frac{\zeta^3}{3} - \frac{\zeta^4}{4}\right]_x^1$

$= \frac{1}{12}(x^4 - x)$

Checking the answer, we can easily verify that y''=x^2, y (0) =0 and y (1) =0.

C. Series Representation of Green's Functions

There are times that it might not be so simple to find the Green's function in the simple closed form. However, there is a method for determining the Green's functions of Sturm-Liouville boundary value problems in the form of an eigenfunction expansion. The eigenfunctions of the differential operator(L), satisfying the homogeneous boundary conditions:

$L[y] = -\lambda_n \sigma \varphi_n$, n = 1, 2,……..

We want to find the particular solution y satisfying L[y] = f and homogeneous boundary conditions. We assume that

$$y(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$$

Inserting this into the differential equation, we obtain

$L[y] = \sum_{n=1}^{\infty} a_n L[\varphi_n] = -\sum_{n=1}^{\infty} \lambda_n a_n \sigma \varphi_n = f$

This has resulted in the generalized Fourier expansion

$$f(x) = \sum_{n=1}^{\infty} c_n \sigma\varphi_n(x)$$

with coefficients

$$c_n = -\lambda_n a_n$$

we have seen how to compute these coefficients earlier.

We multiply both sides by $\varphi_k(x)$ and integrate. Using the orthogonality of the eigenfunctions,

$$\int_a^b \varphi_n(x)\, \varphi_k(x)\, \sigma(x)\, dx = N_k\delta_{nk}$$

one obtains the expansion coefficients (if $\lambda_k \neq 0$)

$$a_k = \frac{-(f,\varphi_n)}{N_n\lambda_n}$$

Where $(f, \varphi_k) \equiv \int_a^b f(x)\, \varphi_n(x)\, dx$

We can rearrange the solution to obtain the Green's function namely, we have

$$y(x) = \sum_{n=1}^{\infty} \frac{(f,\varphi_n)}{-N_n\lambda_n} \varphi_a(x) = \int_a^b \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\zeta)}{-N_n\lambda_n} f(\zeta)\, d\zeta$$

Therefore, we have found the Green's function as an expansion in the eigenfunctions:

$$G(x,\zeta) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\zeta)}{-N_n\lambda_n}$$

2) Example.

Solve y” + 4y = x^2, x ∈ (0, 1), y (0) =y (1) = 0 using the Green’s function as an eigenfunction expansion.

The Green’s function for this problem can be constructed fairly quickly for this problem once the eigenvalue problem is solved.

The eigenvalue problem is

$$\varphi''(x) + 4\varphi(x) = -\lambda\, \varphi(x)$$

Where $\varphi(0) = 0$ and $\varphi(1) = 0$

The general solution is obtained by rewriting the equation as

$$\varphi''(x) + k^2\varphi(x) = 0, \text{ Where } K^2 = 4 + \lambda$$

Solutions satisfying the boundary condition at x=0 are of the form

$$\varphi(x) = A \sin kx$$

Forcing $\varphi(1)$=0 gives

$$0 = A \sin k \Rightarrow k = n\pi,\ k = 1, 2, 3\ldots\ldots\ldots$$

So, the eigenvalues are

$$\lambda_n = n^2\pi^2 - 4,\ n=1, 2\ldots\ldots\ldots\ldots$$

And the eigenfunctions are

$$\varphi_n = \sin n\pi x,\ n=1, 2\ldots\ldots\ldots\ldots$$

We also need the normalization constant, N_n. we have that

$$N_n = ||\,\varphi_n\,||^2 = \int_0^1 \sin^2 n\pi x = \frac{1}{2}$$

We can now construct the Green’s function for this problem using the equation

$$G(x, \zeta) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\varphi_n(\zeta)}{-N_n\lambda_n}$$

$$G(x, \zeta) = 2\ \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\zeta}{(4-n^2\pi^2)}$$

Using this Green’s function, the solution of the boundary value problem becomes

$$y(x) = \int_0^1 G(x,\zeta)\, f(\zeta)\, d\zeta$$

$$= \int_0^1 (2\ \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi\zeta}{(4-n^2\pi^2)})\zeta^2 d\zeta$$

$$= 2\ \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \int_0^1 \zeta^2 \sin n\pi\zeta\ d\zeta$$

$$= 2\ \sum_{n=1}^{\infty} \frac{\sin n\pi x}{(4-n^2\pi^2)} \left[\frac{(2-n^2\pi^2)(-1)^n - 2}{n^3\pi^3}\right]$$

III. CONCLUSION

The Green’s function for Sturm – Liouville problem is determined. The green’s function for ordinary differential equation is obtained for series representation.

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