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Bivariate Interpolation in Triangular Form

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ABSTRACT

The purpose of this paper is to derive lagrange interpolation formula for a single variable and two independent variables in triangular form. Firstly, single variable interpolation is derived and then two independent variable interpolation derived in triangular form. In this paper, we derived the formula of three points of fit approximation, six points of fit approximation and ten points of fit approximation with their examples respectively. And also derived the approximation using a general triangular form of points with examples.
Keywords: Three-points-of-fit, six-points-of –fit and ten-points-of-fit, general triangular form of points, interpolation, approximation.

I. INTRODUCTION

Interpolation, in mathematics, is a curve fitting method. Given a set of data points, we use interpolation techniques to fit different types of curves that pass through the given data points. We will see how to perform linear interpolation on a pair of data points in 2 dimensions.

Consider an unknown function whose y values are y_0 and y_1 at the x values x_0 and x_1 respectively. To fit a linear curve that passes through the two data points given, we simply need to find the equation of the straight line that passes through the two points. We can do this by two data points given, we simply need to find the equation of the straight line that passes through the two points. We can do this by using the two-point form of the equation of a straight line.

$$\frac{y-y_0}{x-x_0}=\frac{y_1-y_0}{x_1-x_0}$$

$$y=y_0+\left(\frac{y_1-y_0}{x_1-x_0}\right)\times\left(x-x_0\right)$$

The liner function $y=f(x)$ described above is known as the linear interpolation for the given two data points. [2]

Bivariate interpolation based an univariate subdivision by N Sharon. The bivariate subdivision scheme interpolating data consisting of univariate functions along equidistant parallel lines by repeated refinements solving the bivariate interpolation problem. The speed of our algorithm is determined by the speed of the method to solve the different univariate interpolation.[2]

The Bilinear Interpolation and Bivariate Interpolation commands are used for the first type. The bilinear interpolation is analogous to linear interpolation. The bivariate interpolation uses an interpolating function that is a piecewise polynomial function that is represented as a tensor product of one-dimensional B-splines. That is,where U(i) and V(j) are one-dimensional B-spline basis functions and the coefficients a(i,j) are chosen so that the interpolating function equals the z axis input values at the grid points. The main point of this paper to perform a bivariate interpolation of a series of gridded data points.[2]

II. BIVARIATE APPROXIMATION IN TRIANGULAR FORMS

For the same input dataset, in order to avoid having to calculate the natural neighbor relationships every time interpolation is done on a single point, interpolation at single points is implemented as a three step process. Do the interpolation at the desired points.[2]

First of all, we will review the following result. Let f(x) be a function of two variables x and y with known values $f(x_0) = f_0$ and $f(x_1) = f_1$, where $x_0 \neq x_1$, which are situated in the form of a right triangle. Suppose x is a point so that

$$\min\{x_0, x_1\} \le x \le \max\{x_0, x_1\}.$$

$$f(x) \approx y(x) = Ax + B$$

with $y(x_0) = f_o, y(x_1) = f_1, x_0 \le x \le x_1$

$$y(x_0) = Ax_0 + B = f_0 \;\dots\dots\dots (1)$$
$$y(x_1) = Ax_1 + B = f_1 \;\dots\dots\dots (2)$$

$$A(x_0 - x_1) = f_0 - f_1$$

$$A = \frac{f_0 - f_1}{x_0 - x_1}$$

$$(1) \Rightarrow \quad \frac{f_0x_0 - f_1x_0}{x_0 - x_1} + B = f_0$$

$$B = f_0 - \frac{f_0x_0 - f_1x_0}{x_0 - x_1}$$

$$B = \frac{f_0x_0 - f_0x_1 - f_0x_0 + f_1x_0}{x_0 - x_1}$$

$$B = \frac{f_1x_0 - f_0x_1}{x_0 - x_1}$$

$$y(x) = Ax + B$$

$$y = \frac{f_0x - f_1x}{x_0 - x_1} + \frac{f_1x_0 - f_0x_1}{x_0 - x_1}$$

$$y = \frac{f_0x - f_1x + f_1x_0 - f_0x_1}{x_0 - x_1}$$

$$y = \frac{(x - x_1)f_0}{x_0 - x_1} + \frac{(x_0 - x)f_1}{x_0 - x_1}$$

$$y = \frac{(x - x_1)f_0}{x_0 - x_1} - \frac{(x - x_0)f_1}{x_0 - x_1}$$

$$y = \frac{(x - x_1)f_0}{x_0 - x_1} + \frac{(x - x_0)f_1}{x_1 - x_0}$$

A. Three-Points-of-Fit

Suppose (x, y) lie interior to the triangle formed by three points, namely (x_0, y_0), (x_0, y_1) and (x_1, y_0). We approximate f(x, y) by using known values: $f(x_0, y_0) = f_{00}$, $f(x_0, y_1) = f_{01}$ and $f(x_1, y_0) = f_{10}$ in the following manner.

Then

$$f(x,y) \approx \left(\frac{x - x_0}{x_0 - x_1} + \frac{y - y_0}{y_0 - y_1} + 1\right) f_{00} + \left(\frac{y - y_0}{y_1 - y_0}\right) f_{01} + \left(\frac{x - x_0}{x_0 - x_1}\right) f_{10}$$

for $x - x_0 = uh$, $x_1 - x_0 = h$, $y - y_0 = vk$,

$y_1 - y_0 = k$,

$$f(x, y) \approx f_{00} + u\,\Delta_x f_{00} + v\,\Delta_y f_{00}$$

where $\Delta_x f_{00} = f_{10} - f_{00}$, $\Delta_y f_{00} = f_{01} - f_{00}$.

Proof

Let $f(x, y) \approx z(x, y) = Ax + By + C$ satisfying

$$A x_0 + B y_0 + C = f_{00}, \dots\dots\dots\dots (1)$$
$$A x_0 + B y_1 + C = f_{01}, \dots\dots\dots\dots (2)$$
$$A x_1 + B y_0 + C = f_{10} \dots\dots\dots\dots (3)$$

By subtracting (1) from (2),

$$B(y_1 - y_0) = f_{01} - f_{00}$$
$$B = \frac{f_{01} - f_{00}}{y_1 - y_0}.$$

By subtracting (1) from (3),

$$A(x_1 - x_0) = f_{10} - f_{00}$$
$$A = \frac{f_{10} - f_{00}}{x_1 - x_0}.$$

By (1), $\quad C = f_{00} - A x_0 - B y_0$

$$= f_{00} - \left(\frac{f_{10}-f_{00}}{x_1-x_0}\right)x_0 - \left(\frac{f_{01}-f_{00}}{y_1-y_0}\right)y_0$$

$$= \left(1+\frac{x_0}{x_1-x_0}+\frac{y_0}{y_1-y_0}\right)f_{00} - \left(\frac{y_0}{y_1-y_0}\right)f_{01} - \left(\frac{x_0}{x_1-x_0}\right)f_{10}.$$

$$z(x, y) = Ax + By + C$$

$$z(x, y) = \left(\frac{f_{10}-f_{00}}{x_1-x_0}\right)x + \left(\frac{f_{01}-f_{00}}{y_1-y_0}\right)y + \left(1+\frac{x_0}{x_1-x_0}+\frac{y_0}{y_1-y_0}\right)f_{00} - \left(\frac{y_0}{y_1-y_0}\right)f_{01} - \left(\frac{x_0}{x_1-x_0}\right)f_{10}$$

$$= \left(1+\frac{x_0}{x_1-x_0}+\frac{y_0}{y_1-y_0}-\frac{x}{x_1-x_0}-\frac{y}{y_1-y_0}\right)f_{00} + \left(\frac{y}{y_1-y_0}-\frac{y_0}{y_1-y_0}\right)f_{01} + \left(\frac{x}{x_1-x_0}-\frac{x_0}{x_1-x_0}\right)f_{10}$$

$$= \left(\frac{x-x_0}{x_0-x_1}+\frac{y-y_0}{y_0-y_1}+1\right)f_{00} + \left(\frac{y-y_0}{y_1-y_0}\right)f_{01} + \left(\frac{x-x_0}{x_1-x_0}\right)f_{10}.$$

For $x - x_0 = uh$, $x_1 - x_0 = h$, $y - y_0 = vk$, $y_1 - y_0 = k$,

$$f(x, y) \approx f_{00} + u\,\Delta_x f_{00} + v\,\Delta_y f_{00}$$

where $\Delta_x f_{00} = f_{10} - f_{00}$, $\Delta_y f_{00} = f_{01} - f_{00}$.

1) Example

Let $x_0 = 0.4$, $x_1 = 0.7$, $y_0 = 0$ and $y_1 = 0.05$ with respective known values $f_{00} = 2.5$, $f_{01} = 2.487$ and $f_{10} = 1.429$. We will approximate f(0.5, 0.03). Then we compare with its true value 2.074. We use Bivariate Approximation with Three-points-of-fit.

Now, $x = 0.5$ and $y = 0.03$.

$$f(x, y) \approx \left(\frac{x-x_0}{x_0-x_1}+\frac{y-y_0}{y_0-y_1}+1\right)f_{00} + \left(\frac{y-y_0}{y_1-y_0}\right)f_{01} + \left(\frac{x-x_0}{x_1-x_0}\right)f_{10}$$

$$= \left[\frac{0.5-0.4}{0.4-0.7}+\frac{0.03-0}{0-0.05}+1\right](2.5) + \left[\frac{0.03-0}{0.05-0}\right](2.487) + \left[\frac{0.5-0.4}{0.7-0.4}\right](1.429)$$

$$= \left[\frac{0.1}{-0.3}+\frac{0.03}{-0.05}+1\right](2.5) + \left[\frac{0.03}{0.05}\right](2.487) + \left[\frac{0.1}{0.3}\right](1.429)$$

$= (0.0667)(2.5)+(0.6)(2.487)+(0.3333)(1.429)$

$= 0.1668+1.4922+0.4763$

$= 2.1353$

Since $2.074 - 2.1353 = -0.0613$ it is true for one decimal place.

B. Six-Points-of-Fit Approximation

Suppose (x, y) lie interior to the triangle formed by six points, namely (x_0, y_0), (x_0, y_1), (x_0, y_2), (x_1, y_0), (x_1, y_1) and (x_2, y_0). We approximate f(x, y) by using known values: $f(x_0, y_0) = f_{00}$, $f(x_0, y_1) = f_{01}$, $f(x_0, y_2) = f_{02}$, $f(x_1, y_0) = f_{10}$, $f(x_1, y_1) = f_{11}$ and $f(x_2, y_0) = f_{20}$ in the following manner.

For $x - x_0 = uh$, $x_1 - x_0 = x_2 - x_1 = h$,

$y - y_0 = vk$, $y_1 - y_0 = y_2 - y_1 = k$,

$$f(x, y) \approx f_{00} + u\,\Delta_x f_{00} + v\,\Delta_y f_{00} + \frac{u(u-1)}{2}\Delta^2_{xx} f_{00} + uv\,\Delta^2_{xy} f_{00} + \frac{v(v-1)}{2}\Delta^2_{yy} f_{00}$$

where $\Delta^2_{xx} f_{00} = f_{20} - 2f_{10} + f_{00}$,

$\Delta^2_{xy} f_{00} = f_{11} - f_{10} - f_{01} + f_{00}$,

$\Delta^2_{yy} f_{00} = f_{02} - 2f_{01} + f_{00}$.

Proof: Let $f(x, y) \approx z(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$ satisfying

$$Ax_0^2 + Bx_0y_0 + Cy_0^2 + Dx_0 + Ey_0 + F = f_{00}, \quad \text{……….} (1)$$

$$Ax_0^2 + Bx_0y_1 + Cy_1^2 + Dx_0 + Ey_1 + F = f_{01}, \quad \text{……….} (2)$$

$$Ax_0^2 + Bx_0y_2 + Cy_2^2 + Dx_0 + Ey_2 + F = f_{02}, \quad \text{……….} (3)$$

$$Ax_1^2 + Bx_1y_0 + Cy_0^2 + Dx_1 + Ey_0 + F = f_{10}, \quad \text{……….} (4)$$

$$Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = f_{11}, \quad \text{……….} (5)$$

$$Ax_2^2 + Bx_2y_0 + Cy_0^2 + Dx_2 + Ey_0 + F = f_{20}. \quad \text{……….} (6)$$

By subtraction (1) from (2),

$$Bx_0(y_1 - y_0) + C(y_1^2 - y_0^2) + E(y_1 - y_0) = f_{01} - f_{00}$$

$$Bx_0 + C(y_1 + y_0) + E = \frac{f_{01} - f_{00}}{y_1 - y_0} \quad \text{……….} (7)$$

By subtraction (1) from (3),

$$Bx_0(y_2 - y_0) + C(y_2^2 - y_0^2) + E(y_2 - y_0) = f_{02} - f_{00}$$

$$Bx_0 + C(y_2 + y_0) + E = \frac{f_{02} - f_{00}}{y_2 - y_0} \quad \text{……….} (8)$$

By Subtraction (7) from (8),

$$C(y_2 - y_1) = \frac{f_{02} - f_{00}}{y_2 - y_0} - \frac{f_{01} - f_{00}}{y_1 - y_0}$$

$$C = \frac{f_{02} - f_{00}}{(y_2 - y_0)(y_2 - y_1)} - \frac{f_{01} - f_{00}}{(y_1 - y_0)(y_2 - y_1)}$$

$$= \frac{y_1f_{02} - y_0f_{02} - y_1f_{00} + y_0f_{00} - y_2f_{01} + y_0f_{01} + y_2f_{00} - y_0f_{00}}{(y_1 - y_0)(y_2 - y_0)(y_2 - y_1)}$$

$$= \frac{(y_2 - y_1)f_{00} + (y_0 - y_2)f_{01} + (y_1 - y_0)f_{02}}{(y_1 - y_0)(y_2 - y_0)(y_2 - y_1)}.$$

By subtraction (4) from (5),

$$Bx_1(y_1 - y_0) + C(y_1^2 - y_0^2) + E(y_1 - y_0) = f_{11} - f_{10}$$

$$Bx_1 + C(y_1 + y_0) + E = \frac{f_{11} - f_{10}}{y_1 - y_0} \quad \text{……….} (9)$$

By subtraction (7) from (9),

$$B(x_1 - x_0) = \frac{f_{11} - f_{10}}{y_1 - y_0} - \frac{f_{01} - f_{00}}{y_1 - y_0}$$

$$B = \frac{f_{11} - f_{10} - f_{01} + f_{00}}{(x_1 - x_0)(y_1 - y_0)}.$$

By subtracting (1) form (4),

$$A(x_1^2 - x_0^2) + By_0(x_1 - x_0) + D(x_1 - x_0) = f_{10} - f_{00}$$

$$A(x_1 + x_0) + By_0 + D = \frac{f_{10} - f_{00}}{x_1 - x_0} \quad \text{……….} (10)$$

By subtracting (1) form (6),

$$A(x_2^2 - x_0^2) + By_0(x_2 - x_0) + D(x_2 - x_0) = f_{20} - f_{00}$$

$$A(x_2 + x_0) + By_0 + D = \frac{f_{20} - f_{00}}{x_2 - x_0} \quad \text{……….} (11)$$

By subtracting (10) from (11),

$$A(x_2 - x_1) = \frac{f_{20} - f_{00}}{x_2 - x_0} - \frac{f_{10} - f_{00}}{x_1 - x_0}$$

$$A = \frac{f_{20} - f_{00}}{(x_2 - x_0)(x_2 - x_1)} - \frac{f_{10} - f_{00}}{(x_1 - x_0)(x_2 - x_1)}$$

$$= \frac{x_1f_{20} + x_0f_{00} - x_0f_{20} - x_1f_{00} - x_2f_{10} + x_0f_{10} + x_2f_{00} - x_0f_{00}}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{(x_2 - x_1)f_{00} + (x_0 - x_2)f_{10} + (x_1 - x_0)f_{20}}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}.$$

By (10),

$$D = \frac{f_{10} - f_{00}}{x_1 - x_0} - A(x_1 + x_0) - By_0$$

$$= \frac{f_{10} - f_{00}}{x_1 - x_0} - \left(\frac{(x_2 - x_1)f_{00} + (x_0 - x_2)f_{10} + (x_1 - x_0)f_{20}}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}\right)(x + x_0) - \left(\frac{f_{11} - f_{10} - f_{01} + f_{00}}{(x_1 - x_0)(y_1 - y_0)}\right)y_0.$$

By (9),

$$E = \frac{f_{11} - f_{10}}{y_1 - y_0} - C(y_1 + y_0) - Bx_1$$

$$= \frac{f_{11} - f_{10}}{y_1 - y_0} - \left(\frac{(y_2 - y_1)f_{00} + (y_0 - y_2)f_{01} + (y_1 - y_0)f_{02}}{(y_1 - y_0)(y_2 - y_0)(y_2 - y_1)}\right)(y_1 + y_0) - \left(\frac{f_{11} - f_{10} - f_{01} + f_{00}}{(x_1 - x_0)(y_1 - y_0)}\right)x_1.$$

By (1),

$$F = f_{00} - Ax_0^2 - Bx_0y_0 - Cy_0^2 - Dx_0 - Ey_0$$

$$= f_{00} - \frac{(x_2 - x_1)f_{00} + (x_0 - x_2)f_{10} + (x_1 - x_0)f_{20}}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}x_0^2 - \left(\frac{f_{11} - f_{10} - f_{01} + f_{00}}{(x_1 - x_0)(y_1 - y_0)}\right)x_0y_0 - \left(\frac{(y_2 - y_1)f_{00} + (y_0 - y_2)f_{01} + (y_1 - y_0)f_{02}}{(y_1 - y_0)(y_2 - y_0)(y_2 - y_1)}\right)y_0^2$$

$$-\left[\frac{f_{10} - f_{00}}{x_1 - x_0} - \left(\frac{(x_2 - x_1)f_{00} + (x_0 - x_2)f_{10} + (x_1 - x_0)f_{20}}{(x_1 - x_0)(x_2 - x_0)(x_2 - x_1)}\right)(x + x_0)\right.$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)y_0\Bigg]x_0 - \Bigg[\frac{f_{11}-f_{10}}{y_1-y_0}$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right)(y_1+y_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_1\Bigg]y_0 .$$

$$z(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$$

$$= \left[\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\right]x^2$$

$$+\left[\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right]xy$$

$$+\left[\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right]y^2$$

$$+\Bigg[\frac{f_{10}-f_{00}}{x_1-x_0}$$

$$-\left(\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\right)(x+x_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)y_0\Bigg]x + \Bigg[\frac{f_{11}-f_{10}}{y_1-y_0}$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right)(y_1+y_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_1\Bigg]y + f_{00}$$

$$- \frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\ x_0^2$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_0 y_0$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right)y_0^2$$

$$- \Bigg[\frac{f_{10}-f_{00}}{x_1-x_0}$$

$$- \left(\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\right)(x+x_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)y_0\Bigg]x_0 - \Bigg[\frac{f_{11}-f_{10}}{y_1-y_0}$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right)(y_1+y_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_1\Bigg]y_0 .$$

For $x - x_0 = uh$, $x_1 - x_0 = x_2 - x_1 = h$, $y - y_0 = vk$, $y_1 - y_0 = y_2 - y_1 = k$,

$$f(x, y) \approx f_{00} + u\,\Delta_x f_{00} + v\,\Delta_y f_{00} + \frac{u(u-1)}{2}\Delta^2_{xx} f_{00}$$

$$+ uv\,\Delta^2_{xy} f_{00} + \frac{v(v-1)}{2}\Delta^2_{yy} f_{00}$$

where $\Delta^2_{xx} f_{00} = f_{20} - 2 f_{10} + f_{00}$,

$\Delta^2_{xy} f_{00} = f_{11} - f_{10} - f_{01} + f_{00}$,

$\Delta^2_{yy} f_{00} = f_{02} - 2 f_{01} + f_{00}$.

1) Example

Let $x_0 = 0.4$, $x_1 = 0.7$, $x_2 = 1$, $y_0 = 0$, $y_1 = 0.05$ and $y_2=0.1$ with respective known values $f_{00} = 2.5$, $f_{01} = 2.487$, $f_{02}=2.456$, $f_{10} = 1.429$, $f_{11} = 1.419$ and $f_{20} = 0.995$. We will approximate f(0.5, 0.03). Then we compare with its true value 2.074. We use Bivariate Approximation with Six-points-of-fit.

Now, $x = 0.5$ and $y = 0.03$.

$$f(x,y) \approx \left[\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\right]x^2$$

$$+\left[\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right]xy$$

$$+\left[\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right]y^2$$

$$+\Bigg[\frac{f_{10}-f_{00}}{x_1-x_0}$$

$$-\left(\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\right)(x+x_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)y_0\Bigg]x + \Bigg[\frac{f_{11}-f_{10}}{y_1-y_0}$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)}\right)(y_1+y_0)$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_1\Bigg]y + f_{00}$$

$$- \frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)}\ x_0^2$$

$$- \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)}\right)x_0 y_0$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)} \right) y_0^2$$

$$- \left[\frac{f_{10}-f_{00}}{x_1-x_0} \right.$$

$$- \left(\frac{(x_2-x_1)f_{00}+(x_0-x_2)f_{10}+(x_1-x_0)f_{20}}{(x_1-x_0)(x_2-x_0)(x_2-x_1)} \right)(x+x_0)$$

$$\left. - \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)} \right) y_0 \right] x_0 - \left[\frac{f_{11}-f_{10}}{y_1-y_0} \right.$$

$$- \left(\frac{(y_2-y_1)f_{00}+(y_0-y_2)f_{01}+(y_1-y_0)f_{02}}{(y_1-y_0)(y_2-y_0)(y_2-y_1)} \right)(y_1+y_0)$$

$$\left. - \left(\frac{f_{11}-f_{10}-f_{01}+f_{00}}{(x_1-x_0)(y_1-y_0)} \right) x_1 \right] y_0 .$$

$$= \left[\frac{(1-0.7)(2.5)+(0.4-1)(1.429)+(0.7-0.4)(0.995)}{(0.7-0.4)(1-0.4)(1-0.7)} \right](0.5)^2 +$$

$$\left[\frac{1.419-1.429-2.487+2.5}{(0.7-0.4)(0.05-0)} \right](0.5)(0.03) +$$

$$\left[\frac{(0.1-0.05)(2.5)+(0-0.1)(2.487)+(0.05-0)(2.456)}{(0.05-0)(0.1-0)(0.1-0.05)} \right](0.03)^2 +$$

$$\left[\frac{1.429-2.5}{0.7-0.4} \right.$$

$$- \left(\frac{(1-0.7)(2.5)+(0.4-1)(1.429)+(0.7-0.4)(0.995)}{(0.7-0.4)(1-0.4)(1-0.7)} \right)(0.9)$$

$$\left. - \left(\frac{1.419-1.429-2.487+2.5}{(0.7-0.4)(0.05-0)} \right)(0) \right](0.5)$$

$$+ \left[\frac{1.419-1.429}{0.05-0} \right.$$

$$- \left(\frac{(0.1-0.05)(2.5)+(0-0.1)(2.487)+(0.05-0)(2.456)}{(0.05-0)(0.1-0)(0.1-0.05)} \right)$$

$$\left. (0.05+0) \right](0.03)$$

$$- \left[\left(\frac{1.419-1.429-2.487+2.5}{(0.7-0.4)(0.05-0)} \right)(0.7) \right](0.03)$$

$$+(2.5)$$

$$- \left(\frac{(1-0.7)(2.5)+(0.4-1)(1.429)+(0.7-0.4)(0.995)}{(0.7-0.4)(1-0.4)(1-0.7)} \right)(0.4)^2$$

$$- \left(\frac{1.419-1.429-2.487+2.5}{(0.7-0.4)(0.05-0)} \right)(0.4)(0)$$

$$- \left(\frac{(0.1-0.05)(2.5)+(0-0.1)(2.487)+(0.05-0)(2.456)}{(0.05-0)(0.1-0)(0.1-0.05)} \right)(0)$$

$$- \left[\frac{1.429-2.5}{0.7-0.4} \right.$$

$$- \left(\frac{(1-0.7)(2.5)+(0.4-1)(1.429)+(0.7-0.4)(0.995)}{(0.7-0.4)(1-0.4)(1-0.7)} \right)(0.9)$$

$$\left. - \left(\frac{1.419-1.429-2.487+2.5}{(0.7-0.4)(0.05-0)} \right)(0) \right](0.4)$$

$$= 0.8847 + 0.0003 - 0.0003 - 3.3775 - 0.0005 + 1.9338 + 2.702 - 0$$

$$= 2.1425$$

Since $2.074 - 2.1425 = -0.0685$ it is true for one decimal place.

C. Ten-Points of Fit Approximation

Suppose (x, y) lie interior to the triangle formed by ten points, namely (x_0, y_0), (x_0, y_1), (x_0, y_2), (x_0, y_3), (x_1, y_0), (x_1, y_1), (x_1, y_2), (x_2, y_0), (x_2, y_1) and (x_3, y_0). We approximate f(x, y) by using known values: $f(x_0, y_0) = f_{00}$, $f(x_0, y_1) = f_{01}$, $f(x_0, y_2) = f_{02}$, $f(x_0, y_3) = f_{03}$, $f(x_1, y_0) = f_{10}$, $f(x_1, y_1) = f_{11}$, $f(x_1, y_2) = f_{12}$, $f(x_2, y_0) = f_{20}$, $f(x_2, y_1) = f_{21}$ and $f(x_3, y_0) = f_{30}$ in the following manner.

For $x - x_0 = uh$, $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = h$, $y - y_0 = vk$, $y_1 - y_0 = y_2 - y_1 = y_3 - y_2 = k$,

$$f(x, y) \approx f_{00} + u\,\Delta_x f_{00} + v\,\Delta_y f_{00} + \frac{u(u-1)}{2}\Delta_{xx}^2 f_{00}$$
$$+ uv\,\Delta_{xy}^2 f_{00} + \frac{v(v-1)}{2}\Delta_{yy}^2 f_{00}$$
$$+ \frac{u(u-1)(u-2)}{6}\Delta_{xxx}^3 f_{00}$$
$$+ \frac{u(u-1)v}{2}\Delta_{xxy}^3 f_{00} + \frac{uv(v-1)}{2}\Delta_{xyy}^3 f_{00}$$
$$+ \frac{v(v-1)(v-2)}{6}\Delta_{yyy}^3 f_{00}$$

where $\Delta_{xxx}^3 f_{00} = f_{30} - 3f_{20} - 3f_{10} + f_{00}$,

$\Delta_{xxy}^3 f_{00} = f_{21} - 2f_{11} + 2f_{10} + f_{01} - 2f_{00}$,

$\Delta_{xyy}^3 f_{00} = f_{12} - f_{02} - 2f_{11} + 2f_{01} + f_{01} - f_{00}$.

$\Delta_{yyy}^3 f_{00} = f_{03} - 3f_{02} - 3f_{01} + f_{00}$.

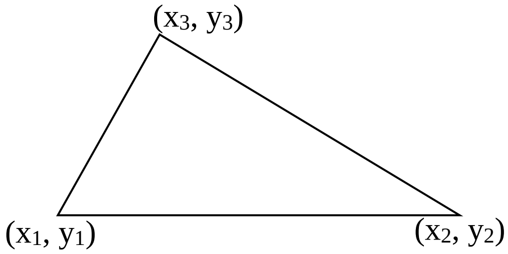
Proof

Let $f(x, y) \approx z(x, y) = Ax^3 + By^3 + Cx^2y + Dxy^2 + Ex^2 + Fy^2 + Gxy + Hx + Iy + J$.

It can be obtained by using argument like previous proofs.

D. Approximation Using a General Triangular Form of Points

We consider (x_1, y_1), (x_2, y_2) and (x_3, y_3) where $x_2 - x_1 \neq x_3 - x_2$ and $y_2 - y_1 \neq y_3 - y_2$.



Let f be a function of variables x and y satisfying $f(x_1, y_1) = f_{11}$, $f(x_2, y_2) = f_{22}$ and $f(x_3, y_3) = f_{33}$.

Suppose (x, y) is a point interior to the triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) as shown in figure.

We approximate f(x, y) with $z(x, y) = A + Bx + Cy$ satisfying $z(x_1, y_1) = f_{11}$, $z(x_2, y_2) = f_{22}$ and $z(x_3, y_3) = f_{33}$.

Then
$$A + Bx_1 + Cy_1 = f_{11},$$
$$A + Bx_2 + Cy_2 = f_{22} \text{ and}$$
$$A + Bx_3 + Cy_3 = f_{33}.$$

For non-collinearity of (x_1, y_1), (x_2, y_2) and (x_3, y_3) the coefficient determinant

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \neq 0.$$

Hence the system has a unique solution for (A, B, C). The system is equivalent to

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} f_{11} \\ f_{22} \\ f_{33} \end{pmatrix}$$

and

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} f_{11} \\ f_{22} \\ f_{33} \end{pmatrix}.$$

Since
$$z(x, y) = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix},$$

$$z(x, y) = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} f_{11} \\ f_{22} \\ f_{33} \end{pmatrix}.$$

Set
$$N = \begin{pmatrix} N_1 & N_2 & N_3 \end{pmatrix} = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1}.$$

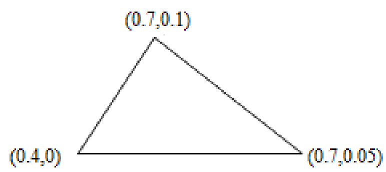
Then

$$(N_1, N_2, N_3) = \begin{pmatrix} 1 & x & y \end{pmatrix} \frac{1}{2\alpha} \begin{pmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{pmatrix},$$

where α = area of the triangle described above.

1) Example

Let $(x_1, y_1) = (0.4, 0)$, $(x_2, y_2) = (0.7, 0.05)$ and $(x_3, y_3) = (0.7, 0.1)$ where $x_2 - x_1 \neq x_3 - x_2$ and $y_2 - y_1 \neq y_3 - y_2$.



Let f be a function of variables x and y satisfying f(0.4, 0) = 0, f(0.7, 0.05) = 0.05 and f(0.7, 0.1) = 0.075.

Suppose (x, y) = (0.5, 0.075) is a point interior to the triangle with vertices (0.4,0), (0.7, 0.05) and (0.7, 0.1) as shown in figure.

We approximate f(0.5,0.075) with z(0.5, 0.075) = A + 0.5B + 0.075C satisfying z(0.4, 0) = 0, z(0.7, 0.05) = 0.05 and z(0.7,0.1) = 0.075.

Then $A + 0.4B + (0)C = 0$

$$A + 0.7B + 0.05C = 0.05 \text{ and}$$

$$A + 0.7B + 0.1C = 0.075$$

For non-collinearity of (0.4, 0), (0.7, 0.05) and (0.7, 0.1) the coefficient determinant

$$\begin{vmatrix} 1 & 0.4 & 0 \\ 1 & 0.7 & 0.05 \\ 1 & 0.7 & 0.1 \end{vmatrix} = 1\begin{vmatrix} 0.7 & 0.05 \\ 0.7 & 0.1 \end{vmatrix} - 0.4\begin{vmatrix} 1 & 0.05 \\ 1 & 0.1 \end{vmatrix} + 0\begin{vmatrix} 1 & 0.7 \\ 1 & 0.7 \end{vmatrix}$$

$$= 0.035-0.02+0$$

$$= 0.015$$

$$\neq 0.$$

And then

$$\begin{bmatrix} 1 & 0.4 & 0 \\ 1 & 0.7 & 0.05 \\ 1 & 0.7 & 0.1 \end{bmatrix}^{-1}$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0.4 & 0 & 1 & 0 & 0 \\ 1 & 0.7 & 0.05 & 0 & 1 & 0 \\ 1 & 0.7 & 0.1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_1(-1)+R_2 \\ \\ R_1(-1)+R_3 \end{matrix}$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0.4 & 0 & 1 & 0 & 0 \\ 0 & 0.3 & 0.05 & -1 & 1 & 0 \\ 0 & 0.3 & 0.1 & -1 & 0 & 1 \end{array}\right] (\frac{1}{0.3})R_2$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0.4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0.17 & -3.33 & 3.33 & 0 \\ 0 & 0.3 & 0.1 & -1 & 0 & 1 \end{array}\right] \begin{matrix} R_2(-0.4)+R_1 \\ \\ R_2(-0.3)+R_3 \end{matrix}$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0 & -0.068 & 1 & 0 & -0.068 \\ 0 & 1 & 0.17 & 0 & 1 & 0.17 \\ 0 & 0 & 0.049 & 0 & 0 & 0.049 \end{array}\right] (\frac{1}{0.049})R_3$$

$$: \left[\begin{array}{ccc|ccc} 1 & 0 & -0.068 & 2.332 & -1.332 & 0 \\ 0 & 1 & 0.17 & -3.33 & 3.33 & 0 \\ 0 & 0 & 1 & 0 & -20 & 20 \end{array}\right] \begin{matrix} R_3(0.068)+R_1 \\ \\ R_3(-0.17)+R_2 \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2.332 & -2.692 & 1.36 \\ 0 & 1 & 0 & -3.33 & 6.73 & -3.4 \\ 0 & 0 & 1 & 0 & -20 & 20 \end{array}\right]$$

$$= \begin{bmatrix} 2.33 & -2.67 & 1.33 \\ -3.33 & 6.67 & -3.33 \\ 0 & -20 & 20 \end{bmatrix}$$

Hence the system has a unique solution for (A, B, C). The system is equivalent to

$$\begin{bmatrix} 1 & 0.4 & 0 \\ 1 & 0.7 & 0.05 \\ 1 & 0.7 & 0.1 \end{bmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{bmatrix} 0 \\ 0.05 \\ 0.075 \end{bmatrix}$$

and

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{bmatrix} 2.33 & -2.67 & 1.33 \\ -3.33 & 6.67 & -3.33 \\ 0 & -20 & 20 \end{bmatrix} \begin{bmatrix} 0 \\ 0.05 \\ 0.075 \end{bmatrix}$$

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{bmatrix} -1.233 \\ 3.083 \\ -8.5 \end{bmatrix}$$

Since $z(0.5,0.075) = \begin{pmatrix} 1 & 0.5 & 0.075 \end{pmatrix} \begin{bmatrix} -1.233 \\ 3.083 \\ -8.5 \end{bmatrix}$

$$= 0.3289,$$

Set $N = \begin{pmatrix} N_1 & N_2 & N_3 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0.5 & 0.075 \end{pmatrix} \begin{bmatrix} 2.33 & -2.67 & 1.33 \\ -3.33 & 6.67 & -3.33 \\ 0 & -20 & 20 \end{bmatrix}.$$

Then

(N_1, N_2, N_3)

$$= \begin{pmatrix} 1 & 0.5 & 0.075 \end{pmatrix} \frac{1}{2\alpha} \begin{bmatrix} (0.7)(0.1)-(0.7)(0.05) & (0.)-(0.4)(0.1) & (0.4)(0.05)-(0) \\ (0.05-0.1) & (0.01-0) & (0-0.05) \\ (0.7-0.7) & (0.4-0.7) & (0.7-0.4) \end{bmatrix}$$

where α = area of the triangle described above. Hence

$$N_1 = \frac{1}{0.015}\Big\{\big((0.7)(0.1)-(0.7)(0.05)\big)$$
$$+0.5(0.05-0.1)+0.075(0.7-0.7)\Big\},$$

$$N_2 = \frac{1}{0.015}\Big\{\big((0.7)(0)-(0.4)(0.1)\big)$$
$$+0.5(0.1-0)+0.075(0.4-0.7)\Big\} \text{ and}$$

$$N_3 = \frac{1}{0.015}\Big\{\big((0.4)(0.5)-(0.7)(0)\big)$$
$$+0.5(0-0.05)+0.075(0.7-0.4)\Big\}.$$

For convenience of notation, we may write,

$$N_1 = \frac{1}{0.015}(a_1 + 0.5b_1 + 0.075c_1),$$

$$N_2 = \frac{1}{0.015}(a_2 + 0.5b_2 + 0.075c_2),$$

$$N_3 = \frac{1}{0.015}(a_3 + 0.5b_3 + 0.075c_3),$$

where $a_1 = (0.7)(0.1) - (0.7)(0.05),$

$a_2 = (0.7)(0) - (0.4)(0.1),$

$a_3 = (0.4)(0.05) - (0.7)(0),$

$b_1 = 0.05 - 0.1$, $b_2 = 0.1 - 0$, $b_3 = 0 - 0.05$,

$c_1 = 0.7 - 0.7$, $c_2 = 0.4 - 0.7$,

$c_3 = 0.7 - 0.4$.

Thus,

$$z(0.5,0.075) = (0.66)(0) + (-0.83)(0.05) + (1.17)(0.075) = 0.0463,$$

$$x = (0.66)(0.4) + (-0.83)(0.7) + (1.17)(0.7) = 0.5,$$

$$y = (0.66)(0) + (-0.83)(0.05) + (1.17)(0.1) = 0.075.$$

$$N_1 + N_2 + N_3 = 0.66 - 0.83 + 1.17 = 1.$$

III. CONCLUSION

It has been presented about bivariate interpolation formulae starting with three- points - of - fit , six-points-of-fit and ten-point-of-fit . Also showed approximation using a general triangular form of points. Some practical problems has been solved by these interpolation formula respectively.

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