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# Fractional Differentiation and Wavelet Analysis

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**ABSTRACT**

In general we know that there are many basis of $L^2(R)$ space, in which some can be written in the terms of $sin(x)$ and $cos(x)$. Another way of producing an orthonormal basis from single function involves translations and modulation. In this paper we discuss to construction an orthonormal basis of $L^2(R)$ from given basis with help of fractional differentiation which is different from parent basis.

**Keywords:** Inner Product, Fourier Transformation, Orthonormal Basis, Fractional Differentiation.

**2010 Mathematics subject classification:** 42C40, 46C15,42A38,13N99.

## I. INTRODUCTION

### 1.1 Fractional Calculus

Fractional differentiation is generalization of differentiation and integration of any (non integeral) order.Concept of fractional differentiation is came in existence about three hundred years of history,in 1965 Leibniz wrote a letter to L'Hospital raising the following question?

"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hopital was somewhat curious about the above question and replied by another simple one to Leibniz: ?What if the order will be 1/2? Leibniz in a letter dated September 30,1695 , replied: ?It will lead to a paradox, from which one day useful consequences will be drawn.

That date is birthday of fraction calculus.in 1730 best mathematical minds in Europe, Euler,wrote;
"When n is a positive integer and p is a function of x,p=p(x), the ratio of $d^n p$ to $d^n x$ can n always be expressed algebraically. But what kind of ratio can then be made if n be a fraction"?

Many mathematician contribute to development of fractional calculus as Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890,but in 1819 Lacroix [6], gave the correct answer to the problem raised by Leibnitz and L?Hospital for the first time, that

$$\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = 2\sqrt{\frac{x}{\pi}} \tag{1.1}$$

The fractional derivative cannot be analytically computed except some special functions,such as following see Refs [6, 4, 1]:

$$\frac{d^{\nu}}{dx^{\nu}}cos(x) = cos(x+\nu\frac{\pi}{2}); \frac{d^{\nu}}{dx^{\nu}}sin(ax) = a^{\nu}sin(ax+\frac{\pi}{2}\nu); \\ \frac{d^{\nu}}{dx^{\nu}}e^{ax} = a^{\nu}e^{ax}; \tag{1.2}$$

where $\nu$ is any real number.

### 1.2 Wavelet Analysis

Wavelet analysis is nothing but the classical windowed of Fourier analysis.Fourier analysis,windowed Fourier analysis,and wavelet analysis based on an identical recipe.The synthesis is obtained exactly as if these building block ware an orthonormal basis.The wavelet analysis came in existence to fulfill drawback of standard Fourier series expansions in pure mathematics.when any one measuring the size or smoothness of function arise the difficulty as,the simplest norm,based on quadratic estimates,can easily estimated estimated Fourier coefficients.But as soon as $L^p$ $or$ $H^p$ estimates are addresses,Fourier coefficient do not answer the problem.The correct tools that give solution of problem was Harr basis (1909),the Franklin orthonormal system (1927) or Littlewood -paley theory (1930). Later Calderon's reproducing identity (1960) and atomic decompositions (1972) were widely use in other functional setting (hard space).D.Gabor (1946) introduced time-frequency atoms in speech signal processing and in 1975 Esteban,and Galand developed subband coding in signal processing;after little time Burnt and Adelson described pyramidal algorithms in image processing in (1982).[12]

There are many basis of $L^2(R)$ some of them given by .Let $g = \chi_{[0,1]}$ and translation and dilation of g is $g_{m,n}(x) = e^{\pi 2imx}g(x-n)\ for\ m,n \in Z$ .We can see that $g_{n,m}: m,n \in Z$ is an orthonormal basis for $L^2(R)$.There are other example of orthonormal basis given as,

$$\begin{cases} a)\ \ \sqrt{2}sine(\frac{2k+1}{2}\pi\ x); k = 0,1,2,3,4.\ldots.; \\ b)\ \ \sqrt{2}sine(\pi\ k\ x); k = 1,2,3,4,\ldots.,; \\ c)\ \ \sqrt{2}cos(\frac{2k+1}{2}\pi\ x); k = 0,1,2,3,4.\ldots.,; \\ d)\ \ \sqrt{2}cos(\pi\ k\ x),1; k = 1,2,3,4.\ldots,;. \end{cases} \tag{1.3}$$

each one of the system is orthonormal basis of $L^2([0,1])$.

## II. Preliminaries

2.1 Definition and properties of Fractional Differential Operator

There are some definition of fractional differentiation of function.Riemann-Liouville fractional order derivative defined as [6],

$$\frac{d^{\nu}}{dx^{\nu}}f(x) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dx}\int_0^x \frac{f(\xi)}{(x-\xi)^{\nu}}d\xi; \tag{2.1}$$

, where $f(x)$ is single real value function.

The next definition given by Caputo (for a differentiable function)

$$\frac{d^{\nu}}{dx^{\nu}}f(x)=\frac{1}{\Gamma(1-\nu)}\int_0^x\frac{f'(\xi)}{(x-\xi)^{\nu}}d(\xi). \tag{2.2}$$

Where $0<\nu<1$,by Grunwald [4][1]

$$\frac{d^{\nu}}{dx^{\nu}}f(x)=\lim_{N\to\infty}\frac{1}{\Gamma(-\nu)}\left(\frac{x}{N}\right)^{-\nu}\sum_{k=0}^{N-1}\frac{\Gamma(k-\nu)}{\Gamma(k+1)}f\left[\left(1-\frac{k}{N}\right)x\right] \tag{2.3}$$

Where $0<\nu<1, x>0$

The drawback of Gruwald and Riemann-Liouville definition, that it cannot computed for negative value of $x$. The Fourier transformation of a function$f\in L^1(R)\cap, L^2(R)$ is define by,

$$\hat{f}(\xi)=\int\ f\,e^{-\xi\,i\,x} \tag{2.4}$$

Inner product of two function defined as,

$$<f,g>=\int\ f\bar{g} \tag{2.5}$$

**Definition 2.1** *We say that two function $f$ and $g$ are orthogonal when inner product is zero $<f,g>=0$. A sequence of function $\{f_n\}, n\in Z$ is an orthonormal sequence if ,$\langle f_m\ f_n\rangle=\delta_{m,n}$. Where,*

$$\begin{cases}(a)\ \ \delta_{m,n}=1\ if\ m=n,\\(b)\ \ \delta_{m,n}=0\ if\ m\neq n.\end{cases} \tag{2.6}$$

The orthogonality properties of scaling and wavelet function given by following equation,[3];

$$\begin{aligned}&\langle\psi_k^n(x),\psi_h^m(x)\rangle=\delta^{nm}\delta_{hk},\\&\left\langle\phi_k^0(x),\phi_h^0(x)\right\rangle=\delta_{hk},\\&\left\langle\phi_k^0(x),\psi_h^m(x)\right\rangle=0, m>=0\end{aligned} \tag{2.7}$$

where $\delta^{nm},\delta_{hk}$ are Kronecker delta symbols.

## 2.2 Basic Theorem Properties of Scaling and Wavelet Function

Fourier transformation of derivative of function;

$$\hat{f}^n(\xi)=(i\xi)^n\hat{f}(\xi) \tag{2.8}$$

**Theorem 2.2** ***Ballian Low Theorem:****[12] Suppose $g\in L^2(R)$ and $g_{m,n}(x)=e^{imkx\pi}g(x-n); m,n\in Z$ if $g_{m,n}; m,n\in Z$ is orthonormal basis of$L^2(R)$,Then either$\int_{-\infty}^{\infty}x^2|g(x)|^2dx=\infty$ or $\int_{-\infty}^{\infty}\xi^2|\hat{g}(\xi)|^2d\xi=\infty$*

Some theorem on operator theory given by following Theorem,

**Theorem 2.3** *A linear operator is bounded if and only if, it is continuous.*

**Theorem 2.4** *Suppose* $f \in L^1(-\pi,\pi)$, *Then the Fourier coefficients* $\hat{f}_k$ *converges to zero as* $|k| \to 0$.

To proof of theorem (2.3), (2.4) see Refs[13]

**A multiresolution analysis(MRA)**:

A multiresolution analysis (MRA)[11] consist of a sequence of closed subspaces$V_j, j \in Z$ of $L^2(R)$. Which satisfy the following properties.

$$\begin{cases} a) & V_j \subset V_{j+1} \; all \; j \in Z, \\ b) & f \in V_j \; if \; and \; only \; if \; f(2(.)) \in V_{j+1} \; all \; j \in Z, \\ c) & \bigcap_{j\in Z} V_j = \{0\}, \\ d) & \overline{\bigcup}_{j\in Z} V_j = L^2(R), \\ e) & There \; exist \; a \; function \; \phi \in V_0 \; such \; that \; \{\psi(.-k): k \in Z\} \\ is \; an \; orthonormal \; basis \; for \; V_0. \end{cases} \tag{2.9}$$

**Fractional derivative of scaling and wavelet,[10]**

Now we define fractional order derivative of scaling and wavelet basis with help of linear interpolation of integer order derivative as;[3]

$$\frac{d^{l+\nu}}{dx^{l+\nu}}\phi_h(x); \frac{d^{l+\nu}}{dx^{l+\nu}}\psi_h^m(x) \tag{2.10}$$

where $0 \le \nu \le 1$

The fractional derivative of scaling function and wavelet function Defined as,

$$\frac{d^{l+\nu}}{dx^{l+\nu}}\phi_h(x) = (1-\nu)\frac{d^l}{dx^l}\phi_h(x) + \nu\frac{d^{l+1}}{dx^{l+1}}\phi_h(x). \tag{2.11}$$

$$\frac{d^{l+\nu}}{dx^{l+\nu}}\psi_h^m(x) = (1-\nu)\frac{d^l}{dx^l}\psi_h^m(x) + \nu\frac{d^{l+1}}{dx^{l+1}}\psi_h^m(x). \tag{2.12}$$

## III. Discussion

Take a sequence of function describe in introduction as $cos(\frac{2k+1}{2}\pi\, x)$; $k = 0,1,2,3,\ldots \; in \, L^2([0,1])$ then $g_k(x) = cos(\frac{2k+1}{2}\pi\, x)$ so $g(x) = g_0(x) = g(\frac{\pi}{2}x)$; Now consider a particular case $k = 0$[14] . Then

$$\frac{d^\nu}{dx^\nu}cos(\frac{\pi\, x}{2}) = cos(\frac{\pi\, x}{2} + \frac{\nu\pi}{2}) = g^\nu(\frac{\pi\, x}{2}). \tag{3.1}$$

The fourier transformation of$g^\nu(x)$given by equation,

$$\hat{g}^\nu(\xi) = \int_{-\infty}^{\infty} g^\nu(x)e^{-\xi x i}dx = \int_0^1 cos(\frac{x\pi}{2} + \frac{\nu\pi}{2})dx. \tag{3.2}$$

$$\int_0^1 \frac{1}{2}\left[e^{i}v\frac{\pi}{2}e^{i(\frac{\pi}{2}x-\xi)} + e^{-iv\frac{\pi}{2}6}e^{-i(\frac{\pi}{2}x-\xi)}\right]dx = \frac{1}{i(\pi-2\xi)}\left[e^{i(\frac{\pi}{2}-\xi)}-1\right]e^{iv\frac{\pi}{2}} + \frac{1}{-i(\pi+2\xi)}\left[e^{-i(\frac{\pi}{2}+\xi)}-1\right]e^{-iv\frac{\pi}{2}}. \quad (3.3)$$

After simplification the above equation given by following equation(3.4)

$$\left[\frac{1}{i(\pi-2\xi)}\right]\left[e^{i(\frac{\pi}{2}-\xi)}-1\right]e^{iv\frac{\pi}{2}} + \frac{1}{-i(\pi+2\xi)}\left[e^{-i(\frac{\pi}{2}+\xi)}-1\right]e^{-iv\frac{\pi}{2}} = \frac{2\pi\, e^{i(v\frac{\pi}{2}-\xi)}}{\pi^2-4\xi^2} + \frac{2cos(\frac{v+1}{2}\pi)+i2sine(\frac{v+1}{2}\pi)}{\pi^2-4\xi^2}. \quad (3.4)$$

This gives

$$\lim_{\xi\to\infty}|\hat{g}^v(\xi)|^2\xi^2 = 0. \quad (3.5)$$

Now we will check the second condition of Theorem (2.2).

$$\int_{-\infty}^{\infty}|g^v(x)|^2x^2dx = 2\int_0^{\infty}|g^v(x)|^2x^2dx. \quad (3.6)$$

$$2\int_0^{\infty}|g^v(x)|^2x^2dx = \lim_{c\to\infty}\int_0^{c}|g^v(x)|^2x^2dx. \quad (3.7)$$

by equation[1] we have

$$\lim_{c\to\infty}\int_0^{c}\left|cos\left(\frac{\pi}{2}x+v\frac{\pi}{2}\right)\right|^2x^2dx \to \infty. \quad (3.8)$$

This is particular case for $k = 1$ the function $g_{k=0}^{v}(x)$ satisfy the Balian Low Theorem.Next we analyze for any integer k,

$$\frac{d^v}{dx^v}cos\left(\frac{2k+1}{2}\pi x\right) = cos\left(\frac{2k+1}{2}\pi x + v\frac{\pi}{2}\right). \quad (3.9)$$

$$\hat{g}_k^v(\xi) = \int_{-\infty}^{\infty}g_k^v(x)e^{-i\xi x}dx. \quad (3.10)$$

$$\int_{-\infty}^{\infty}g_k^v(x)e^{-i\xi x}dx = \int_0^1\frac{1}{2}(e^{i}\left((2k+1)\frac{\pi}{2}+v\frac{\pi}{2}\right) + e^{\left(-i(2k+1)\frac{\pi}{2}+v\frac{\pi}{2}\right)})e^{-i\xi x}. \quad (3.11)$$

$$\int_0^1\frac{1}{2}(e^{i\left((2k+1)\frac{\pi}{2}+v\frac{\pi}{2}\right)} + e^{-i\left\{(2k+1)\frac{\pi}{2}+v\frac{\pi}{2}\right\}})e^{-i\xi x}dx = \frac{2(2k+1)\pi}{(2k+1)^2\pi^2-4\xi^2} \times [cos\left\{(2k+1)\frac{\pi}{2}+\xi\right\} + icos(v\frac{\pi}{2})] + \frac{2\xi}{(2k+1)^2\pi^2-4\xi^2} \times \left[sine\left\{(2k+1)\frac{\pi}{2}-\xi\right\} - i\,sine\left(v\frac{\pi}{2}\right)\right]. \quad (3.12)$$

We will test the condition of Balian Low Theorem on the function $g_k^v(x)$, it being orthonormal basis.

$$\int_{-\infty}^{\infty}|g_k^v(x)|^2x^2dx = \int_{-\infty}^{\infty}\left|cos\left((2k+1)\frac{\pi}{2}x+v\frac{\pi}{2}\right)\right|^2x^2dx = \infty; when\ x\to\infty. \quad (3.13)$$

and

$$\int_{-\infty}^{\infty}|\hat{g}_k^v(\xi)|^2\xi^2d\xi = 0; When\ \xi\to\infty. \quad (3.14)$$

Now we test the condition of multiresolution analysis on scaling and wavelet function derived from fractional differentiation.In the light of equation (2.7) orthogonality condition of function does not change with differentiation.

The differentiation of a function is an operator, it follow the theorem (2.3) and (2.4). So the convergence of function represented by equation (2.11) and (2.12) are valid.

**Theorem 3.1** *The fractional differential operator forms semi group;[3]*

$$\frac{d^{\mu}}{dx^{\mu}}\frac{d^{\nu}}{dx^{\nu}}\phi_h(x) = \frac{d^{\nu}}{dx^{\nu}}\frac{d^{\mu}}{dx^{\mu}}\phi_h(x) = \frac{d^{\mu+\nu}}{dx^{\mu+\nu}}\phi_h(x). \quad (3.15)$$

Hence any order of fractional differentiation of scaling function $\phi_{k\ or\ h}$ satisfy multiresolution condition so we find the result construction of wavelet from multiresolution analysis.This new construction of wavelet give fast convergence of function comparison to wavelet constructed through scaling function$\phi_{h\ or\ k}$.

## IV.CONCLUSION

Now we have been arrived at the point that when the orthonormal basis of $L^2(R)$ or $L^2[0,1]$ is form another basis of wavelet function.Then we can derived an orthonormal basis using fractional differentiation which give the result in equation (2.11) and (2.12) .When $\nu$ is any integer then it is obvious that it give the translation of given basis, so it will not give the different from the given basis.But when we consider $\nu$ is fractional real number then it gives the basis different from parent basis that can be seen help of equation (2.11) and (1.12) in reference of equation (3.13). There is an interesting thing that the equation that can be find by equation number (2.10) which give scaling function and wavelet function that fulfill the condition of MRA .The equation (2.10) basically give linearly independent function derived from given scaling function . So this method to generate the basis can be powerful technique to find the orthonormal basis for specific work in scientific purpose.

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