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Growth Analysis of Composite Entire Functions of Several Complex Variables Based on Relative Order

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ABSTRACT

In this paper we introduce some new results depending on the comparative growth properties of composition of entire function of several complex variables using relative $L^* -$ order, Relative $L^* -$ lower order and $L \equiv L(r_1, r_2, r_3, \dots \dots .., r_n)$ is a slowly changing functions. We prove some relation between relative $L^* -$ order and relative $L^* -$ lower order.

Keywords: Entire functions, Several complex variables, Maximum modulus, comparative growth, relative order $L^* -$ order , ralative $L^* -$ lower order , Maximum term.

I. INTRODUCTION, DEFINITIONS AND NOTATIONS

In 2007 Banerjee and Datta [6] introduced the definition of relative order of an entire function $f(z_1, z_2)$ with respect to an entire function $g(z_1, z_2)$ as follows.

Let $g(z_1, z_2)$ be an entire function holomorphic in closed polydisc $\{(z_1, z_2): |z_j| \le r_j\,; j = 1,2\}$ and let

$$G(r_1, r_2) = \max\{g(z_1, z_2);\ |z_j| \le r_j\,; j = 1,2\}$$

The relative order of f with respect to g , denoted by $\rho_g(f)$ and is denoted by

$$\rho_g(f) = \inf\{\mu > 0: F(r_1, r_2) < G(r_1^{\mu}, r_2^{\mu}); \text{for } r_1 > R(\mu), r_2 \ge R(\mu)\}.$$

In this paper we introduced the idea of relative order of entire functions of several complex variables.

Note: Subscript v_n denote n variables.

Definition 1. The order ${}^{v_n}\rho_f$ and lower order ${}^{v_n}\lambda_f$ of an entire function f are defined as

$${}^{v_n}\rho_f = \lim_{r_1, r_2, \dots\dots, r_n \to \infty} \sup \frac{\log^{[2]} {}^{v_n}M_f}{\log(r_1.r_2 \dots\dots r_n)} \quad \text{and} \quad {}^{v_n}\lambda_f = \lim_{r_1, r_2, \dots\dots, r_n \to \infty} \inf \frac{\log^{[2]} {}^{v_n}M_f}{\log(r_1.r_2 \dots\dots r_n)}$$

using the inequalities , ${}^{v_n}\mu_f = {}^{v_n}M_f \le \frac{R_1}{R_1 - r_1} \frac{R_2}{R_2 - r_2} \dots \frac{R_n}{R_n - r_n} {}^{v_n}M_f(R)$

{cf. [2]}, for $0 \le r_1 < R_1, 0 \le r_2 < R_2 \dots \dots, 0 \le r_n < R_n$,

$${}^{v_n}\rho_f = \lim_{r_1, r_2, \dots\dots, r_n \to \infty} \sup \frac{\log^{[2]} {}^{v_n}M_f}{\log(r_1.r_2 \dots\dots r_n)} \quad \text{and} \quad {}^{v_n}\lambda_f = \lim_{r_1, r_2, \dots\dots, r_n \to \infty} \inf \frac{\log^{[2]} {}^{v_n}M_f}{\log(r_1.r_2 \dots\dots r_n)}$$

Let $L \equiv L(r_1, r_2, r_3, \dots .., r_n)$ be a positive continuous function slowly increasing

i.e., $$L(ar_1, ar_2, \dots .., ar_n) \sim L(r_1, r_2, \dots .., r_n)$$

as $r_1, r_2, \dots .., r_n \to \infty$ for every positive constant a. Singh and Barker [24] defined in the following way.

Definition 2. [24] A positive continuous function $L(r_1, r_2, \dots .., r_n)$ is called a slowly changing function. If

$$\frac{1}{k^{\varepsilon}} \le \frac{L(kr_1, kr_2, \ldots\ldots, kr_n)}{L(r_1, r_2, \ldots\ldots, r_n)} \le k^{\varepsilon}, \quad \text{for } r_1, r_2, \ldots\ldots, r_n \ge r(\varepsilon) \text{ and uniformly for } k(\ge 1).$$

The notion of $L-$ order and $L-$ lower order for entire functions, introduce the Somasundaram and Thamizharasi in [7]

The more generalized concept for $L-$ order and $L-$ lower order for entire functions are L^*- order, and L^*- lower order respectively. Their definitions are as follows:

Definition 3. [7] The L^*- order, ${}^{v_n}\rho_f^{L^*}$ and L^*- lower order ${}^{v_n}\lambda_f^{L^*}$ of an entire function f are defined as

$${}^{v_n}\rho_f^{L^*} = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \sup \frac{\log^{[2]} {}^{v_n}M_f(r)}{\log[(r_1.r_2 \ldots\ldots r_n)\, e^{L(r_1, r_2, \ldots\ldots, r_n)}]} \text{ and}$$

$${}^{v_n}\lambda_f^{L^*} = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \inf \frac{\log^{[2]} {}^{v_n}M_f(r)}{\log[(r_1.r_2 \ldots\ldots r_n)\, e^{L(r_1, r_2, \ldots\ldots, r_n)}]}$$

If an entire function g is non-constant then $M_g(r)$ is strictly increasing and continuous. Its inverse ${}^{v_n}M_g^{-1}: (|f(0,0, \ldots\ldots, 0)|, \infty) \to (0, \infty)$, exists and is constant

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} {}^{v_n}M_g^{-1}(r) = \infty$$

Banerjee and Datta [6] introduced the definition of relative order of an entire function f of several complex variables w.r.t. an entire function of several complex variables g, denoted by ${}^{v_n}\rho_g(f)$ as follows

$${}^{v_n}\rho_g(f) = \inf\left\{\mu > 0;\ {}^{v_n}M_f(r) < {}^{v_n}M_g(r^{\mu}), \text{for all } r_1, r_2, \ldots\ldots, r_n > r_0(\mu) > 0\right\}$$

$$= \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \sup \frac{\log {}^{v_n}M_g^{-1}\, {}^{v_n}M_f(r)}{\log(r_1 r_2 \ldots\ldots r_n)}, \quad \text{for } g(z_1, z_2, \ldots\ldots z_n) = \exp(z_1.z_2 \ldots\ldots z_n)$$

The above definition coincides with classical one $\{cf.$ [8]$\}$.

Similarly, we can define the relative lower order of an entire function f of several complex variables with respect to another entire function of several complex variables g denoted by

$${}^{v_n}\lambda_g(f) = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \inf \frac{\log {}^{v_n}M_g^{-1}\, {}^{v_n}M_f(r)}{\log(r_1 r_2 \ldots\ldots r_n)}$$

Datta and Maji [14] gave an another definition of relative order and relative lower order of an entire function of several complex variables with respect to another entire function of several complex variables in this following.

***Definition* 4**. [14] The relative order ${}^{v_n}\rho_g(f)$ and relative lower order ${}^{v_n}\lambda_g(f)$ of an entire function of several complex variables f with respect to another entire function g of several complex variables are defined by

$${}^{v_n}\rho_g(f) = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \sup \frac{\log {}^{v_n}M_g^{-1}\, {}^{v_n}M_f(r)}{\log(r_1 r_2 \ldots\ldots r_n)} \text{ and } {}^{v_n}\lambda_g(f) = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \inf \frac{\log {}^{v_n}M_g^{-1}\, {}^{v_n}M_f(r)}{\log(r_1 r_2 \ldots\ldots r_n)}.$$

***Definition* 5**. ([18][17]) The relative $L^*-order$ of an entire function f of several complex variables with respect to an entire function g of several complex variables defined by

$${}^{v_n}\rho_g^{L^*} = \inf\left\{\mu > 0 :\ {}^{v_n}M_f(r) < M_g\{(r_1.r_2 \ldots r_n)e^{L(r_1, r_2, \ldots, r_n)}\}, for\ all\ r_1, r_2, \ldots, r_n > r_0(\mu) > 0\right\}$$

$$= \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \sup \frac{\log M_g^{-1}\, {}^{v_n}M_f(r)}{\log[(r_1.r_2 \ldots\ldots r_n)\, e^{L(r_1, r_2, \ldots\ldots, r_n)}]}$$

analogously, relative $L^*-lower\ order$ defined by

$${}^{v_n}\lambda_f^{L^*} = \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \inf \frac{\log M_g^{-1}\, {}^{v_n}M_f(r)}{\log[(r_1.r_2 \ldots\ldots r_n)\, e^{L(r_1, r_2, \ldots\ldots, r_n)}]}$$

Datta, Biswas and Ali[19] also gave another definition which as follows:

***Definition* 6**. [19] The relative $L^*-\ order$ ${}^{v_n}\rho_g^{L^*}(f)$ and the relative $L^*-lower\ order$ ${}^{v_n}\lambda_g^{L^*}(f)$ of an entire function f of several complex variables with respect to g are as follows

$$^{v_n}\rho_g^{L^*}(f) = \lim_{r_1,r_2,\dots,r_n\to\infty} sup \frac{log\, M_g^{-1\, v_n} M_f(r)}{log[(r_1.r_2 \dots r_n)\, e^{L(r_1,r_2,\dots,r_n)}]} \quad and$$

$$^{v_n}\lambda_g^{L^*}(f) = \lim_{r_1,r_2,\dots,r_n\to\infty} inf \frac{log\, M_g^{-1\, v_n} M_f(r)}{log[(r_1.r_2 \dots r_n)\, e^{L(r_1,r_2,\dots,r_n)}]}$$

In this paper we study some comparative growth properties of maximum term and maximum modulus of composition of entire functions of several complex variables with respect to another entire function on the basis of relative $L^* - order$ and relative $L^* - lower\ order$. We do not explain the some standard definition and notations in the theory of entire functions of several complex variables as those are available in [9].

In this paper we use some lemmas.

***Lemma* 1**. ([**1**], [**12**]) Let f and g be two entire functions of several complex variables. Then for every $\alpha > 1$ and $0 < r_1 < R_1, 0 < r_2 < R_2, \dots.., 0 < r_n \le R_n$,

$$^{v_n}\mu_{fog}(r) \le \frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha R_1}{R_1 - r_1}.\frac{R_2}{R_2-r_2}\dots\dots..\frac{R_n}{R_n - r_n}\mu_g(r)\right).$$

***Lemma* 2**. ([**1**], [**12**]) Let f and g be any two entire functions of several complex variables with $g(0,0,\dots.0) = 0$. Then for all sufficient large values of $r_1, r_2, \dots\dots, r_n$,

$$^{v_n}\mu_{fog}(r) \ge \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{R_1}{4},\frac{R_2}{4},\dots\dots..,\frac{R_n}{4}\right)\right).$$

***Lemma* 3**. ([**12**], [**10**]) If f and g are any two entire functions of several complex variables then for all sufficient large values of $r_1, r_2, \dots\dots, r_n$,

$$^{v_n}M_{fog}(r_1, r_2, \dots\dots, r_n) = M_f\left(^{v_n}M_g(r)\right).$$

***Lemma* 4**. ([**10**], [**12**]) If f and g are any two entire functions of several complex variables with $g(0) = 0$. Then for all sufficient large values of $r_1, r_2, \dots\dots, r_n$,

$$^{v_n}M_{fog}(r) \ge M_f\left(\frac{1}{8}M_g\left(\frac{r_1}{2},\frac{r_2}{2},\dots\dots..,\frac{r_n}{2}\right)\right).$$

***Lemma* 5**. ([**14**], [**12**]) If f be entire functions and $\alpha > 1, 0 < \beta < \alpha$, Then for all sufficient large values of $r_1, r_2, \dots\dots, r_n$.

$$\mu_f(\alpha r_1, \alpha r_2, \dots\dots, \alpha r_n) \ge \beta\ ^{v_n}\mu_f(r).$$

In this section we prove some main results.

Theorem A. Let f and h be any two entire functions of several complex variables with $0 < {}^{v_n}\lambda_h^{L^*}(f) \le {}^{v_n}\rho_h^{L^*}(f) < \infty$. Also let g be an entire function of several complex variables with $^{v_n}\lambda_g^{L^*} > 0\ and\ g(0,0,\dots.0) = 0$. Then for every positive constant P and real number x.

$$\lim_{r_1,r_2,\dots\dots,r_n\to\infty} \frac{log\, \mu_h^{-1\, v_n}\mu_{fog}(r)}{\{log\, \mu_h^{-1}\mu_f\,(r_1^P, r_2^P, \dots\dots, r_n^P)^{1+x}\}} = \infty.$$

Proof. If x is such that $1 + x \le 0$, then the theorem is obvious. So we suppose that $1 + x > 0$. From lemma (2) and lemma (5), we have for all sufficiently large values of $r_1, r_2, \dots\dots, r_n$ that

(**1**) $\quad ^{v_n}\mu_{fog}(r) \ge \mu_r\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4},\frac{r_2}{4},\dots\dots..,\frac{r_n}{4}\right)\right)$

Since μ_h^{-1} is an increasing function, it follows from (1) for all sufficiently large values of $r_1, r_2, \dots\dots, r_n$ i.e.,

$$\mu_h^{-1\, v_n}\mu_{fog}(r) \ge \mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4},\frac{r_2}{4},\dots\dots..,\frac{r_n}{4}\right)\right)$$

$i.e., \quad log\, \mu_h^{-1\, v_n}\mu_{fog}(r) \ge log\, \mu_h^{-1}\mu_f\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4},\frac{r_2}{4},\dots\dots..,\frac{r_n}{4}\right)\right)$

$log\, \mu_h^{-1\, v_n}\mu_{fog}(r) \ge O(1) + \left(^{v_n}\lambda_h^{L^*}(f) - \varepsilon\right)\left[log\left\{\frac{1}{24}\mu_g\left(\frac{r_1}{4},\frac{r_2}{4},\dots\dots..,\frac{r_n}{4}\right)\right\} + L\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4},\frac{r_2}{4},\dots\dots..,\frac{r_n}{4}\right)\right)\right]$

$log\, \mu_h^{-1\, v_n} \mu_{fog}(r) \geq O(1) + \left({}^{v_n}\lambda_h^{L^*}(f) - \varepsilon\right)\left[log\, \mu_g\left(\frac{r_1}{4}, \frac{r_2}{4}, \ldots\ldots, \frac{r_n}{4}\right) + O(1) + L\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4}, \frac{r_2}{4}, \ldots\ldots, \frac{r_n}{4}\right)\right)\right]$

$i.e., (\mathbf{2}) \quad log\, \mu_h^{-1\, v_n} \mu_{fog}(r) \geq O(1) + \left({}^{v_n}\lambda_h^{L^*}(f) - \varepsilon\right)\left\{\frac{(r_1 r_2 \ldots\ldots r_n)}{4^n} e^{L(r_1, r_2, \ldots\ldots, r_n)}\right\}^{{}^{v_n}\lambda_g^{L^*} - \varepsilon} +$
$L\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4}, \frac{r_2}{4}, \ldots\ldots, \frac{r_n}{4}\right)\right)$, where we can choose $0 < \varepsilon < min\left\{{}^{v_n}\lambda_h^{L^*}(f), {}^{v_n}\lambda_g^{L^*}(f)\right\}$
also for all sufficiently large values of $r_1, r_2, \ldots\ldots, r_n$ .we get that

$$log\, \mu_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P) \leq \left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right) log\left\{(r_1^P r_2^P \ldots\ldots r_n^P)\, e^{L(r_1^P, r_2^P, \ldots\ldots, r_n^P)}\right\}$$

$i.e., \quad log\, \mu_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P) \leq \left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right) log\left\{(r_1^P r_2^P \ldots\ldots r_n^P)\, e^{L(r_1^P, r_2^P, \ldots\ldots, r_n^P)}\right\}$

$i.e., (\mathbf{3}) \quad \left\{log\, \mu_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P)\right\}^{1+x} \leq \left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right)^{1+x} (P\, log\{(r_1^P r_2^P \ldots\ldots r_n^P) +$
$L(r_1^P, r_2^P, \ldots\ldots, r_n^P)\})^{1+x}$

There for (2) and (3), it follows for all sufficiently large values of $r_1, r_2, \ldots\ldots, r_n$ that

$$(\mathbf{4}) \quad \frac{log\, \mu_h^{-1\, v_n} \mu_{fog}(r)}{\left\{log\, \mu_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P)\right\}^{1+x}}$$

$$\geq \frac{O(1) + \left({}^{v_n}\lambda_h^{L^*}(f) - \varepsilon\right)\left\{\frac{r_1 r_2 \ldots\ldots r_n}{4^n} e^{L(r_1, r_2, \ldots\ldots, r_n)}\right\}^{{}^{v_n}\lambda_g^{L^*} - \varepsilon} + L\left(\frac{1}{24}\mu_g\left(\frac{r_1}{4}, \frac{r_2}{4}, \ldots\ldots, \frac{r_n}{4}\right)\right)}{\left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right)^{1+x} (A\, log\{(r_1^P r_2^P \ldots\ldots r_n^P) + L(r_1^P, r_2^P, \ldots\ldots, r_n^P)\})^{1+x}}$$

Since $\frac{r^{{}^{v_n}\lambda_g^{L^*} - \varepsilon}}{log(r_1 r_2 \ldots\ldots r_n)^{1+x}} \to \infty\; as\; r_1, r_2, \ldots\ldots, r_n \to \infty$, the theorem follows from (4)

Theorem (A) state the following theorem without proof.

Theorem B. Let f and h be any two entire functions of several complex variables with $0 < {}^{v_n}\lambda_h^{L^*}(f) < \infty\; or\; 0 < {}^{v_n}\rho_h^{L^*}(f) < \infty$. Also let g be an entire function with ${}^{v_n}\lambda_g^{L^*} > 0\; and\; g(0,0, \ldots .0) = 0$. then for every positive constant P and real number x.

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} sup \frac{log\, \mu_h^{-1\, v_n} \mu_{fog}(r)}{\left\{log\, \mu_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P)^{1+x}\right\}} = \infty.$$

Theorem (A) and theorem (B) and using lemma (4) verify the following two theorems.

Theorem C. Let f and h be any two entire functions of several complex variables with $0 < {}^{v_n}\lambda_h^{L^*}(f) \leq {}^{v_n}\rho_h^{L^*}(f) < \infty$. Also let g be an entire function of several complex variables with ${}^{v_n}\lambda_g^{L^*} > 0\; and\; g(0,0, \ldots .0) = 0$. Then for every positive constant A and real number x.

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \frac{log\, \mu_h^{-1\, v_n} \mu_{fog}(r)}{\left\{log\, \mu_h^{-1} \mu_g\, (r_1^P, r_2^P, \ldots\ldots, r_n^P)^{1+x}\right\}} = \infty.$$

Theorem D. Let f, g and h be any three entire functions of several complex variables g with non zero $L^* - lower\; order, g(0,0, \ldots .0) = 0$ and either $0 \leq {}^{v_n}\lambda_h^{L^*}(f) < \infty$ or $0 \leq {}^{v_n}\rho_h^{L^*}(f) < \infty$. Then for every positive constant P and real number x.

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} sup \frac{log\, M_h^{-1\, v_n} \mu_{fog}(r)}{\left\{log\, M_h^{-1} \mu_f\, (r_1^P, r_2^P, \ldots\ldots, r_n^P)^{1+x}\right\}} = \infty.$$

Theorem E. Let f and h be any two entire functions of several complex variables with $0 < {}^{v_n}\lambda_h^{L^*}(f) \leq {}^{v_n}\rho_h^{L^*}(f) < \infty$ and g be an entire function with non zero $L^* - lower\; order, and\; g(0,0, \ldots .0) = 0$. Then for any positive integer $\alpha\; and\; \beta$,

$$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty} sup \frac{log^{[2]}\mu_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)+{}^{v_n}K(r,\alpha,L)}=\infty,$$

$$v_nK(r,\alpha,L)=\begin{cases}0, if\ (r_1r_2\ldots\ldots r_n)^\beta=O\left\{L\left(exp(exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha))\right)\right\} as\ r_1,r_2,\ldots\ldots,r_n\to\infty.\\ L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))\ \ other\ wise.\end{cases}$$

Proof. Taking $x=0$ and $P=1$ in theorem (A) we obtain for all sufficiently large values of $r_1,r_2,\ldots\ldots,r_n$ and for $k>1$

$$log\,\mu_h^{-1\,v_n}\mu_{fog}(r)>k\,log\,\mu_h^{-1\,v_n}\mu_f(r)$$

$i.e.,\quad \mu_h^{-1\,v_n}\mu_{fog}(r)>\left\{\mu_h^{-1\,v_n}\mu_f(r)\right\}^k$

$i.e.(5)\quad \mu_h^{-1\,v_n}\mu_{fog}(r)>\mu_h^{-1\,v_n}\mu_f(r)$

therefore from (5), we get for all sufficient large values of $r_1,r_2,\ldots\ldots,r_n$ that

$$log\,M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))>\mu_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$$

$i.e.,\ log\,M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right).log\left\{exp\left(exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)\right)expL(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))\right\}$$

$i.e.,\quad log\,M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right).\{exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)+L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))\}$$

$log\,M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right).\left\{exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)\left(1+\frac{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)}\right)\right\}$$

$log^{[2]}M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>O(1)+log\,exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)\ +log\left\{1+\frac{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)}\right\}$$

$log^{[2]}M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>O(1)+(r_1^\alpha.r_2^\alpha\ \ldots..r_n^\alpha)\ +log\left\{1+\frac{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)}\right\}$$

$log^{[2]}M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>O(1)+(r_1r_2\ldots\ldots r_n)^\alpha\ +L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$$

$$-\,log[exp\{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))\}]+log\left\{1+\frac{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{exp(\mu r_1^\alpha r_2^\alpha\ \ldots..r_n^\alpha)}\right\}$$

$log^{[2]}M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>O(1)+(r_1^\alpha r_2^\alpha\ \ldots..r_n^\alpha)\ +L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$$

$$+\,log\left[\frac{1}{exp\{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))\}}+\frac{L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))}{exp\left\{L\left(exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)\right)\right\}exp(r_1^\alpha r_2^\alpha\ \ldots..r_n^\alpha)}\right]$$

$i.e.,\ \ log^{[2]}M_h^{-1}\mu_{fog}(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha)))$

$$>O(1)+\left(r_1^{\alpha-\beta}r_2^{\alpha-\beta}\ \ldots..r_n^{\alpha-\beta}\right)\left(r_1^\beta.r_2^\beta\ \ldots.r_n^\beta\right)\ +L(exp\ (exp(r_1^\alpha,r_2^\alpha,\ldots\ldots,r_n^\alpha))).$$

again we have for all sufficiently large values of $r_1,r_2,\ldots\ldots,r_n$ that

$$log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)\le\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)log\left\{exp\left(r_1^\beta.r_2^\beta\ \ldots..r_n^\beta\right)e^{L\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)}\right\}$$

$$log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)\le\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left\{log\,exp\left(r_1^\beta.r_2^\beta\ \ldots.r_n^\beta\right)+L\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)\right\}$$

$$log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)\le\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left\{\left(r_1^\beta.r_2^\beta\ \ldots.r_n^\beta\right)+L\left(exp\left(r_1^\beta,r_2^\beta,\ldots\ldots,r_n^\beta\right)\right)\right\}$$

$$i.e.,(6)\quad \frac{\log \mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta}.r_2^{\beta}\,....r_n^{\beta}\right)\right)-\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)L\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}{{}^{v_n}\rho_h^{L^*}(f)+\varepsilon}\le r_1^{\beta}r_2^{\beta}\,.....r_n^{\beta}$$

Now from (5) and (6), it follows for all sufficiently large values of $r_1, r_2, \dots\dots, r_n$ that

$$i.e.,(7)\quad \log^{[2]}\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))$$
$$\ge O(1)$$
$$+\left(\frac{r_1^{\alpha-\beta}r_2^{\alpha-\beta}\,.....r_n^{\alpha-\beta}}{{}^{v_n}\rho_h^{L^*}(f)+\varepsilon}\right)\left[\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)\right.$$
$$\left.-\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)L\left(exp\left(r_1^{\beta}.r_2^{\beta}\,....r_n^{\beta}\right)\right)\right]+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha}))).$$

$$i.e.,(8)\quad \frac{\log^{[2]}\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}$$
$$\ge\frac{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))+O(1)}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}$$
$$+\frac{r_1^{(\alpha-\beta)}r_2^{(\alpha-\beta)}\,.....r_n^{(\alpha-\beta)}}{{}^{v_n}\rho_h^{L^*}(f)+\varepsilon}\left\{1-\frac{\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)L\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}\right\}$$

again from (7) we get for all sufficiently large values of $r_1, r_2, \dots\dots, r_n$ that

$$\frac{\log^{[2]}\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}$$
$$\ge\frac{O(1)+\left(r_1^{(\alpha-\beta)}r_2^{(\alpha-\beta)}\,.....r_n^{(\alpha-\beta)}\right)L\,exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}$$
$$+\frac{\left(\frac{r_1^{(\alpha-\beta)}r_2^{(\alpha-\beta)}\,.....r_n^{(\alpha-\beta)}}{{}^{v_n}\rho_h^{L^*}(f)+\varepsilon}\right)\log\mu_h^{-1}\mu_f\left((exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}$$
$$+\frac{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}$$

$$i.e.\quad(9)\quad \frac{\log^{[2]}\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},......,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}\ge\frac{\frac{O(1)+\left(r_1^{(\alpha-\beta)}r_2^{(\alpha-\beta)}\,.....r_n^{(\alpha-\beta)}\right)L\,exp\left(r_1^{\beta},r_2^{\beta},......,r_n^{\beta}\right)}{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}}{\frac{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},......,r_n^{\beta}\right)\right)}{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}}+$$
$$\frac{\frac{r_1^{(\alpha-\beta)}r_2^{(\alpha-\beta)}\,.....r_n^{(\alpha-\beta)}}{{}^{v_n}\rho_h^{L^*}(f)+\varepsilon}}{1+\frac{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},......,r_n^{\beta}\right)\right)}}+\frac{1}{1+\frac{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},......,r_n^{\beta}\right)\right)}{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},......,r_n^{\alpha})))}}$$

Case I. If $r_1^{\beta}r_2^{\beta}\,.....r_n^{\beta}=O\{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))\}$ then it follows from (8) that

$$\lim_{r_1,r_2,......,r_n\to\infty}\frac{\log^{[2]}\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))}{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)}=\infty,$$

Case II. $r_1^{\beta}r_2^{\beta}\,.....r_n^{\beta}\ne O\{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))\}$ then the following two sub cases.

Subcase (a) If $L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},.....,r_n^{\alpha})))=O\left\{\log\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},.....,r_n^{\beta}\right)\right)\right\}$, then we get from (9) that

$$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty} \frac{log^{[2]}\,\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}{log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},\ldots\ldots,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}=\infty,$$

Subcase (b) If $L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))\sim log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},\ldots\ldots,r_n^{\beta}\right)\right)$ then

$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty} \frac{L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}{log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},\ldots\ldots,r_n^{\beta}\right)\right)}=1$ and we obtain from (9)

$$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty} inf \frac{log^{[2]}\,\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}{log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},\ldots\ldots,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}=\infty,$$

combining case I and case II we obtain that

$$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty} \frac{log^{[2]}\,\mu_h^{-1}\mu_{fog}(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}{log\,\mu_h^{-1}\mu_f\left(exp\left(r_1^{\beta},r_2^{\beta},\ldots\ldots,r_n^{\beta}\right)\right)+L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))}=\infty,$$

$$v_nK(r,\alpha,L)=\begin{cases}0, if\ r_1^{\mu}r_2^{\mu}..r_n^{\mu}=O\left\{L\left(exp\left(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})\right)\right)\right\}\\ \qquad as\ r_1,r_2,\ldots\ldots,r_n\to\infty\\ L(exp\,(exp(r_1^{\alpha},r_2^{\alpha},\ldots\ldots,r_n^{\alpha})))\ \ otherwise.\end{cases}$$

this proves the theorem.

Theorem F. Let f , g and h be any three entire functions of several complex variables such that ${}^{v_n}\rho_g^{L^*}<{}^{v_n}\lambda_h^{L^*}(f)\leq{}^{v_n}\rho_h^{L^*}(f)<\infty$. Then for any $\beta>1$,

$$\lim_{r_1,r_2,\ldots\ldots,r_n\to\infty}\frac{log\,\mu_h^{-1}\,{}^{v_n}\mu_{fog}(r)}{log\,\mu_h^{-1}\,{}^{v_n}\mu_f(r).\,{}^{v_n}k(r,g,L)}=0.$$

$$v_nK(r,g;L)=\begin{cases}1, if\ L\left(\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)=0\left\{r_1^{\alpha}r_2^{\alpha}\ldots\ldots r_n^{\alpha}\,e^{\alpha L(r_1,r_2,\ldots\ldots,r_n)}\right\}\\ \qquad as\ r_1,r_2,\ldots\ldots,r_n\to\infty\ and\ for\ some\ \alpha<{}^{v_n}\lambda_h^{L^*}(f)\\ L\left(\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right), otherwise.\end{cases}$$

Proof. Taking $R_1=\beta r_1,R_2=\beta r_2,\ldots\ldots,R_n=\beta r_n$ in lemma (1) and in view of lemma (5) we have for all sufficiently large values $r_1,r_2,\ldots\ldots,r_n$ that

$${}^{v_n}\mu_{fog}(r)\leq\left(\frac{\alpha}{\alpha-1}\right)\mu_f\left(\frac{\alpha\beta}{(\beta-1)}\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)$$

$i.e.,\ \ {}^{v_n}\mu_{fog}(r)\leq\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)$

Since μ_h^{-1} is an increasing function, it follows from above for all sufficiently large values of $r_1,r_2,\ldots\ldots,r_n$ that

$$\mu_h^{-1}\,{}^{v_n}\mu_{fog}(r)\leq\mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)$$

$i.e.,\quad log\,\mu_h^{-1}\,{}^{v_n}\mu_{fog}(r)\leq log\,\mu_h^{-1}\mu_f\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)$

$i.e.,\ \ log\,\mu_h^{-1}\,{}^{v_n}\mu_{fog}(r)\leq\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left[log\,\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)e^{L\left(\frac{2\alpha^2\beta}{(\alpha-1)(\beta-1)}\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)}+O(1)\right]$

$i.e.,(10)\quad log\,\mu_h^{-1}\,{}^{v_n}\mu_{fog}(r)\leq\ \left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left[log\,\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\,L\left(\mu_g(\beta r_1,\beta r_2,\ldots\ldots,\beta r_n)\right)+O(1)\right]$

$i.e.,\quad log\, \mu_h^{-1\, v_n} \mu_{fog}(r)$

$$\leq \left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{(\beta^n r_1 r_2 \ldots . r_n) e^{L(\beta r_1, \beta r_2, \ldots, \beta r_n)}\right\}^{\left({}^{v_n}\rho_g^{L^*}(f)+\varepsilon\right)} + L\left(\mu_g(\beta r_1, \beta r_2, \ldots ., \beta r_n)\right) + O(1)\right]$$

$i.e., (11)\quad log\, \mu_h^{-1\, v_n} \mu_{fog}(r)$

$$\leq \left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{(\beta^n r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right\}^{\left({}^{v_n}\rho_g^{L^*}(f)+\varepsilon\right)} + L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right)\right]$$

also we obtain for all sufficiently large values of $r_1, r_2, \ldots \ldots, r_n$ that

$$log\, \mu_h^{-1\, v_n} \mu(r) \geq \left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right) log\left[(r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right]$$

$i.e.,\quad log\, \mu_h^{-1\, v_n} \mu(r) \geq \left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right) log\left[(r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right]$

$i.e., (12)\quad \mu_h^{-1\, v_n} \mu(r) \geq \left[(r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right]^{\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right)}$

now from (11) and (12) we get for all sufficiently large values of $r_1, r_2, \ldots \ldots, r_n$ that

$$(13)\quad \frac{log\, \mu_h^{-1\, v_n} \mu_{fog}(r)}{\mu_h^{-1\, v_n} \mu(r)} \leq \frac{\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)\left[\left\{(\beta r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right\}^{\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)} + L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right)\right]}{\left[(r_1 r_2 \ldots .. r_n) e^{L(r_1, r_2, \ldots \ldots, r_n)}\right]^{\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right)}}$$

since ${}^{v_n}\rho_h^{L^*} < {}^{v_n}\lambda_h^{L^*}(f)$, we can choose $\varepsilon(>0)$

$(14)\quad {}^{v_n}\rho_g^{L^*} + \varepsilon < {}^{v_n}\lambda_g^{L^*} - \varepsilon.$

Case I. Let $L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right) = O\left\{(r_1^\alpha r_2^\alpha \ldots .. r_n^\alpha)\, e^{\alpha L(r_1, r_2, \ldots \ldots, r_n)}\right\}$

$as\ r_1, r_2, \ldots \ldots, r_n \to \infty\ and\ for\ some\ \alpha < {}^{v_n}\lambda_h^{L^*}(f)$

as $\alpha < {}^{v_n}\lambda_h^{L^*}(f)$ we can choose $\varepsilon(>0)$ in such way

$(15)\quad \alpha < {}^{v_n}\lambda_h^{L^*}(f)$

since $L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right) = O\left\{(r_1^\alpha r_2^\alpha \ldots .. r_n^\alpha)\, e^{\alpha L(r_1, r_2, \ldots \ldots, r_n)}\right\}$

$as\ r_1, r_2, \ldots \ldots, r_n \to \infty\ we\ get\ on\ using$ (15)

$$\frac{L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right)}{(r_1^\alpha r_2^\alpha \ldots .. r_n^\alpha) e^{\alpha L(r_1, r_2, \ldots \ldots, r_n)}} \to 0\ as\ r_1, r_2, \ldots \ldots, r_n \to \infty$$

$$i.e., (16)\quad \frac{L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right)}{\left[(r_1^\alpha r_2^\alpha \ldots .. r_n^\alpha)\, e^{\alpha L(r_1, r_2, \ldots \ldots, r_n)}\right]^{\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right)}} \to 0\ as\ r_1, r_2, \ldots \ldots, r_n \to \infty$$

now in view of (13), (14) and (16) we obtain

$$(17)\quad \lim_{r_1, r_2, \ldots \ldots, r_n \to \infty} \frac{log\, \mu_h^{-1\, v_n} \mu_{fog}(r)}{\mu_h^{-1\, v_n} \mu_f(r)} = 0.$$

Case II. If $L\left(\mu_g(\beta r_1, \beta r_2, \ldots .., \beta r_n)\right) \neq O\left\{(r_1^\alpha r_2^\alpha \ldots .. r_n^\alpha)\, e^{\alpha L(r_1, r_2, \ldots \ldots, r_n)}\right\}$

$as\ r_1, r_2, \ldots\ldots, r_n \to \infty\ and\ for\ some\ \alpha < {}^{v_n}\lambda_h^{L^*}(f)$ then we get from (13) that for a sequence of values of $r_1, r_2, \ldots\ldots, r_n$ tending to infinity.

$$\frac{\log \mu_h^{-1\,v_n} \mu_{fog}(r)}{\mu_h^{-1\,v_n} \mu_f(r). L\left(\mu_g(\beta r_1, \beta r_2, \ldots., \beta r_n)\right)}$$

$$\leq \frac{\left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right)\left[(r_1 r_2 \ldots. r_n) e^{L(r_1, r_2, \ldots, r_n)}\right]^{\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right)}}{\left[(r_1 r_2 \ldots\ldots r_n) e^{L(r_1, r_2, \ldots, r_n)}\right]^{\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right)} . L\left(\mu_g(\beta r_1, \beta r_2, \ldots, \beta r_n)\right)}$$

$$+ \frac{\left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right)}{\left[(r_1 r_2 \ldots\ldots r_n) e^{L(r_1, r_2, \ldots\ldots, r_n)}\right]^{\left({}^{v_n}\lambda_h^{L^*}(f)-\varepsilon\right)}}$$

now using (14) it follows from (17)

(18) $$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \frac{\log \mu_h^{-1\,v_n} \mu_{fog}(r)}{\mu_h^{-1\,v_n} \mu_f(r). L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} = 0.$$

combining (17) and (18) we obtain

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \frac{\log \mu_h^{-1\,v_n} \mu_{fog}(r)}{\mu_h^{-1\,v_n} \mu_f(r). L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} = 0$$

where

$$ {}_{v_n}K(r, g; L) = \begin{cases} 1, if\ L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right) = O\{(r_1^\alpha r_2^\alpha \ldots\ldots r_n^\alpha)\, e^{\alpha L(r_1, r_2, \ldots\ldots, r_n)}\} \\ \quad as\ r_1, r_2, \ldots\ldots, r_n \to \infty\ and\ for\ some\ \alpha < {}^{v_n}\lambda_h^{L^*}(f) \\ \quad L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right), otherwise. \end{cases}$$

thus the theorem is established.

Theorem G. Let f, g and h be any three entire functions of several complex variables such that ${}^{v_n}\rho_h^{L^*}(f) < \infty$, ${}^{v_n}\lambda_h^{L^*}(g) > 0\ and\ {}^{v_n}\rho_g^{L^*} < \infty$. Then for any $\beta > 1$,

(***a***) if $L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right) = 0\left\{\log \mu_h^{-1\,v_n} \mu_g(r)\right\}$ then

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \sup \frac{\log^{[2]} \mu_h^{-1\,v_n} \mu_{fog}(r)}{\log \mu_h^{-1\,v_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} \leq \frac{{}^{v_n}\rho_g^{L^*}}{{}^{v_n}\lambda_h^{L^*}(g)}$$

(***b***) if $\log \mu_h^{-1\,v_n} \mu_g(r) = 0\left\{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)\right\}$ then

$$\lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \frac{\log^{[2]} \mu_h^{-1\,v_n} \mu_{fog}(r)}{\log \mu_h^{-1\,v_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} = 0$$

Proof. Taking $\log\left\{1 + \frac{\log^{[2]} \mu_h^{-1\,v_n} \mu_{fog}(r)}{\log \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)} \sim \frac{O(1) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{\log \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right\}$

we have from (10) for all sufficiently large values of $r_1, r_2, \ldots\ldots, r_n$

$$\log^{[2]} \mu_h^{-1\,v_n} \mu_{fog}(r) \leq \left({}^{v_n}\rho_h^{L^*}(f) + \varepsilon\right). \log \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\left[1 + \frac{O(1) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{\log \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right]$$

$$i.e., \quad log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq log\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right) + log^{[2]}\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n) + log\left[1+\frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right]$$

$$i.e., \quad log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq log\left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right) + \left({}^{v_n}\rho_h^{L^*}(f)+\varepsilon\right) log\left[(r_1 r_2 \ldots\ldots r_n)e^{L(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right] + log\left\{1+\frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right\}$$

$$i.e., \quad log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq log\left({}^{v_n}\rho_h^{L^*}+\varepsilon\right) + \left({}^{v_n}\rho_g^{L^*}(f)+\varepsilon\right) log\left(\beta^n r_1 r_2 \ldots\ldots r_n e^{L(r_1, r_2, \ldots\ldots, r_n)}\right) + log\left\{1+\frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}\right\}$$

$$i.e., \quad log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq O(1) + \left({}^{v_n}\rho_g^{L^*}+\varepsilon\right) + log\{(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n) + L(r_1, r_2, \ldots\ldots, r_n)\} + \frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}$$

$$i.e., (19) \quad log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq O(1) + \left({}^{v_n}\rho_g^{L^*}+\varepsilon\right) + \{log(r_1.r_2 \ldots\ldots r_n) + L(r_1, r_2, \ldots\ldots, r_n)\} + \left({}^{v_n}\rho_g^{L^*}+\varepsilon\right) log\,\beta + \frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}$$

again from the definition of relative $L^* - lower\ order$ of an entire function of several complex variables with respect to another entire function of several complex variables in term of their maximum terms we have for all sufficiently large values of $r_1, r_2, \ldots\ldots, r_n \to \infty$ that

$$i.e., \quad log\,\mu_h^{-1}\,{}^{v_n}\mu_g(r) \geq \left({}^{v_n}\lambda_h^{L^*}(g)-\varepsilon\right) log\left[(r_1 r_2 \ldots\ldots r_n)e^{L(r_1, r_2, \ldots\ldots, r_n)}\right]$$

$$i.e., \quad log\,\mu_h^{-1}\,{}^{v_n}\mu_g(r) \geq \left({}^{v_n}\lambda_h^{L^*}(g)-\varepsilon\right) \left[log\left((r_1 r_2 \ldots\ldots r_n) + L(r_1, r_2, \ldots\ldots, r_n)\right)\right]$$

$$i.e., (20) \quad log(r_1 r_2 \ldots\ldots r_n) + L(r_1, r_2, \ldots\ldots, r_n) \leq \frac{log\,\mu_h^{-1}\,{}^{v_n}\mu_g(r)}{\left({}^{v_n}\lambda_h^{L^*}(g)-\varepsilon\right)}$$

hence from (19) and (20), it follows for all sufficiently large value of $r_1, r_2, \ldots\ldots, r_n$

$$log^{[2]} \mu_h^{-1}\,{}^{v_n}\mu_{fog}(r) \leq O(1) + \left(\frac{{}^{v_n}\rho_g^{L^*}+\varepsilon}{{}^{v_n}\lambda_h^{L^*}(g)-\varepsilon}\right) log\,\mu_h^{-1}\,{}^{v_n}\mu_g(r) + \left({}^{v_n}\rho_g^{L^*}+\varepsilon\right) log\,\beta + \frac{O(1)+L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\,\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}$$

$$i.e., \quad \frac{log^{[2]} \mu_h^{-1\,\nu_n} \mu_{fog}(r)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}$$

$$\leq \frac{O(1) + \left({}^{\nu_n}\rho_g^{L^*} + \varepsilon\right) log\, \beta}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}$$

$$+ \left(\frac{{}^{\nu_n}\rho_g^{L^*} + \varepsilon}{{}^{\nu_n}\lambda_h^{L^*}(g) - \varepsilon}\right) . \frac{log\, \mu_h^{-1\,\nu_n} \mu_g(r)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}$$

$$+ \frac{O(1) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{\left[log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)\right] log\, \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}$$

$$i.e., (\mathbf{21}) \quad \frac{log^{[2]} \mu_h^{-1\,\nu_n} \mu_{fog}(r)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}$$

$$\leq \frac{\dfrac{O(1) + \left({}^{\nu_n}\rho_g^{L^*} + \varepsilon\right) log\, \beta}{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}}{\dfrac{log\, \mu_h^{-1\,\nu_n} \mu_g(r)}{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} + 1} + \frac{\dfrac{{}^{\nu_n}\rho_g^{L^*} + \varepsilon}{{}^{\nu_n}\lambda_h^{L^*}(g) - \varepsilon}}{1 + \dfrac{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r)}}$$

$$+ \frac{1}{\left[1 + \dfrac{log\, \mu_h^{-1\,\nu_n} \mu_g(r)}{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)}\right] log\, \mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)}$$

since $L\left(\mu_g(\beta r_1, \beta r_2, \ldots, \beta r_n)\right) = O\left\{log\, \mu_h^{-1\,\nu_n} \mu_g(r)\right\}$ $as\ r_1, r_2, \ldots, r_n \to \infty\ and\ (\varepsilon > 0)$,
we obtain from (21) that

$$i.e., (\mathbf{22}) \quad \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} sup \frac{log^{[2]} \mu_h^{-1\,\nu_n} \mu_{fog}(r)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} \leq \frac{{}^{\nu_n}\rho_g^{L^*}}{{}^{\nu_n}\lambda_h^{L^*}(g)}$$

again if $log\, \mu_h^{-1\,\nu_n} \mu_g(r) = O\left\{L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)\right\}$ then from (3.21) we get

$$i.e., (\mathbf{23}) \quad \lim_{r_1, r_2, \ldots\ldots, r_n \to \infty} \frac{log^{[2]} \mu_h^{-1\,\nu_n} \mu_{fog}(r)}{log\, \mu_h^{-1\,\nu_n} \mu_g(r) + L\left(\mu_g(\beta r_1, \beta r_2, \ldots\ldots, \beta r_n)\right)} = 0$$

thus from (22) and (23), the theorems ‘are established.

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II. REFERENCES

[1]. A.D. Singh, On maximum term of composite of entire functions, Proc. Nat. Acad. Sci. India., 59(A) (Part I) (1989) (103 - 115).

[2]. A.D. Singh, M.S. Baloria, On maximum modulus and maximum term of composite of entire functions, Indian J.Pure Appl.Math.,22(12) (1991) 1019 – 1026.

[3]. B.A. Fuks, Theory of analytic functions of several complex variables, Moscow,1963.

[4]. C.O. Kiselman, Order and type as measures of growth for convex or entire functions, Proc.Lond.Math.Soc.66(1993),152 – 186.

[5]. C.O. Kiselman, Pluzisubharmonic functions and potential theory in several complex variables a contribution to the book project, Development of Mathematics 1950 – 2000, edited by Jean Panlpier.

[6]. D. Banerjee, R.K. Datta, Relative order of entire functions of two complex variables, International J.Math.Sci.& Engg.Appl.1(2007),141 – 154.

[7]. D. Somasundaram, R. Thamizharasi, A note on the entire functions of L-bounded index and L-type, Indian J.Pure Appl.Math.,19(3) (March 1988),284 – 293.

[8]. E.C. Titch marsh, the Theory of functions, 2nd ed. Oxford University Press, Oxford, 1968.

[9]. G. Valiron, Lectures on the General theory of Integral, Chelsea Publishing Company, 1949.

[10]. J. Clunie, The composition of entire and meromorphic functions, Mathematical Essays dedicated to A. J. Macintyre, hio University Press,(1970) 75 – 92.

[11]. L. Bernal, Order relative de. Crecimiento de functions enteras, Collect.Math. 39(1988) 209 – 229.

[12]. R.K. Datta, Relative order of Entire functions of several complex variables, Mathematici Bechik 65,2(2013),222 – 233.

[13]. S. Halvarsson, Growth properties of entire functions depending on a parameter, Anales Polonici Math.14(1996),71 – 96.

[14]. S.K. Dutta, A.R. Maji, Relative order of entire functions in terms of their maximum terms, Int.Journal of Math.Analysis,5(43) (2011) 2119 – 2126.

[15]. S.K. Dutta, T. Biswas, D.C. Pramanik, on the growth analysis relative to Maximum Terms based Relative order of entire functions, International J. of Math. Sci. & Engg. Appls. (IJMSEA),8(V) (September,2014) 231 – 242.

[16]. S.K. Dutta, T. Biswas, D.C. Pramanik, On relative order and maximum term – related comparative growth rates of entire functions, Journal of Tripura Mathematical Society, 14 (2012),60 – 68.

[17]. S.K. Dutta, S. Kar, On the L-Order of meromorphic functions based on relative sharing, International Journal of Pure and Applied Mathematics (IJPAM) ,56(2009) 43 – 47.

[18]. S.K. Dutta, T. Biswas, Growth of entire functions based on relative order, International Journal of Pure and Applied Mathematics (IJPAM) ,51(2009) 49 – 58.

[19]. S.K. Dutta, T. Biswas, R. Biswas, On relative based growth estimates of entire functions , International J. of Math. Sci. & Engg. Appls. (IJMSEA),7(II)(March,2013), 59 – 67.

[20]. S. K. Dutta, T. Biswas, R. Biswas, Comparative growth properties of composite entire functions in the light of their relative order, The mathematics student, 82(1-4) (2013) 209 – 216.

[21]. S.K. Dutta, T. Biswas, and S. Ali, Growth estimates of composite entire functions based on maximum terms using their relative L-order, Advances in Applied Mathematical Analysis, 7(2012),119 – 134.

[22]. S.K. Dutta, T. Biswas, S. Bhattacharyya, Relative orders and slowly changing functions oriented growth analysis of composite entire functions. Int.J.Non linear Anal. Appl.6(2015),No.2,113 – 126.

[23]. S.K. Dutta, T. Biswas, S. Bhattacharyya, Growth Analysis of composite entire functions relative to slowly changing functions oriented relative order and relative type, Journal of Complex Analysis, 2014, Article ID 503719, 8 pages.

[24]. S.K. Singh, G.P. Barker. Slowly changing functions and their applications, Indian J. Math.,19(1977),1 – 6.

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