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Study of Fractional Analytic Functions and Local Fractional Calculus

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ABSTRACT

In this present paper, the role of fractional analytic function in local fractional calculus is studied. Some important properties and theorems in local fractional calculus are discussed, such as product rule, quotient rule, chain rule, fundamental theorem of local fractional calculus, change of variable, integration by parts and so on. In addition, we propose several examples and formulas of local fractional calculus.

Keywords : Fractional Analytic Function, Local Fractional Calculus, Fundamental Theorem of Local Fractional Calculus, Product Rule, Quotient Rule, Chain Rule, Change of Variable, Integration by Parts.

I. INTRODUCTION

The importance of studying continuous but nowhere differentiable functions was emphasized a long time ago by Perrin, Poincare and others [1-2]. Local fractional calculus is a theory to study continuous non-differentiable functions and describe the mechanical behaviour in fractal space. It has developed for more than twenty years since Kolwankar and Gangal [3-4] used Riemann-Liouville definition of fractional derivative to obtain local fractional derivative operators. The local fractional derivative and integrals was broadly applied in the field of mathematics, applied science and engineering [5-16].

The purpose of the current paper is to study the role of fractional analytic functions in local fractional calculus. In Section II, we first give the definition of local fractional derivative, and we discuss some local fractional derivative properties, for example, product rule, Leibniz rule, quotient rule, chain rule and so on. Furthermore, we introduce some fractional functions and take them as examples. In Section III, the modified Riemann-Liouville fractional derivative of Jumarie type is introduced, and we provide several properties of this type of fractional derivative. In Section IV, we study fractional analytic functions and its role in local fractional derivative. In Section V, we give the definition of fractional Riemann integral (or called local fractional integral), and prove some important theorem such as fundamental theorem of local fractional calculus, change of variable, integration by parts. In Section VI, the conclusion is given.

II. LOCAL FRACTIONAL DERIVATIVE

The purpose of this section is to state the definition and also introduce the properties of local fractional derivative.

Definition 2.1([16]): Let $0 < \alpha < 1$, $(-1)^{\alpha} = -1$, $f: [a, b] \to R$ and $x_0 \in (a, b)$. f is called local α-fractional differentiable at x_0 if $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}$ exists. And the α-fractional derivative of $f(x)$ at x_0 is denoted by

$$f^{(\alpha)}(x_0) = \Gamma(\alpha+1) \cdot \lim_{x \to x_0} \frac{f(x)-f(x_0)}{(x-x_0)^{\alpha}}, \quad (1)$$

where $\Gamma(\ )$ is the gamma function. If f is local α-fractional differentiable at any point in open interval (a, b), then we say that f is a local α-fractional differentiable function on (a, b). In addition, the n-th order local α-fractional derivative of $f(x)$ is denoted by $f^{(n\alpha)}(x) = \left(f^{(\alpha)}\right)\left(f^{(\alpha)}\right) \cdots \left(f^{(\alpha)}\right)(x)$, where n is a positive integer, and $f^{(0)}(x) = f(x)$.

Theorem 2.2: *Assume that* $0 < \alpha < 1$, $(-1)^{\alpha} = -1$ *and* $x_0 \in (a, b)$. *If* $f: [a, b] \to R$ *is local* α*-fractional differentiable at* x_0, *then* f *is continuous at* x_0.

Proof $\lim_{x \to x_0} f(x)$

$$= \lim_{x \to x_0} [f(x) - f(x_0) + f(x_0)]$$

$$= \lim_{x \to x_0} [f(x) - f(x_0)] + f(x_0)$$

$$= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{(x - x_0)^{\alpha}} \cdot \lim_{x \to x_0} (x - x_0)^{\alpha} + f(x_0)$$

$$= \frac{f^{(\alpha)}(x_0)}{\Gamma(\alpha+1)} \cdot 0 + f(x_0)$$

$$= f(x_0).$$

Q.e.d.

Proposition 2.3: *Let* $0 < \alpha < 1$, $(-1)^{\alpha} = -1$, *and* λ *be a real number, If* $f, g: [a, b] \to R$ *are local* α*-fractional differentiable at* $x \in (a, b)$, *then*

(i) $$(f+g)^{(\alpha)}(x) = f^{(\alpha)}(x) + g^{(\alpha)}(x). \quad (2)$$

(ii) $$(f-g)^{(\alpha)}(x) = f^{(\alpha)}(x) - g^{(\alpha)}(x). \quad (3)$$

(iii) $$(\lambda f)^{(\alpha)}(x) = \lambda f^{(\alpha)}(x). \quad (4)$$

Proposition 2.4 (product rule for local fractional derivative):

$$(f \cdot g)^{(\alpha)}(x) = f^{(\alpha)}(x) \cdot g(x) + f(x) \cdot g^{(\alpha)}(x) \quad (5)$$

Proposition 2.5 (Leibniz rule for local fractional derivative):

$$(f \cdot g)^{(n\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{((n-k)\alpha)}(x) \cdot g^{(k\alpha)}(x). \quad (6)$$

Proposition 2.6 (quotient rule for local fractional derivative):

$$\left(\frac{f}{g}\right)^{(\alpha)}(x) = \frac{f^{(\alpha)}(x) \cdot g(x) - f(x) \cdot g^{(\alpha)}(x)}{g^2}. \quad (7)$$

In the following, some fractional functions are introduced.

Definition 2.7 ([22]): The Mittag-Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad (8)$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Definition 2.8 ([17]): $E_{\alpha}(x^{\alpha})$ is called α-fractional exponential function, and the α-fractional cosine and sine function are defined as follows:

$$cos_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \quad (9)$$

and

$$sin_{\alpha}(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}, \quad (10)$$

where $0 < \alpha < 1$, and x is a real variable.

Example 2.9: We have the following local fractional derivative formulas:

$$E_{\alpha}(x^{\alpha})^{(\alpha)} = E_{\alpha}(x^{\alpha}), \quad (11)$$

$$sin_{\alpha}(x^{\alpha})^{(\alpha)} = cos_{\alpha}(x^{\alpha}), \quad (12)$$

and

$$cos_{\alpha}(x^{\alpha})^{(\alpha)} = -sin_{\alpha}(x^{\alpha}). \quad (13)$$

Therefore,

$$\left(sin_{\alpha}(x^{\alpha}) + cos_{\alpha}(x^{\alpha})\right)^{(\alpha)} = cos_{\alpha}(x^{\alpha}) - sin_{\alpha}(x^{\alpha}), \quad (14)$$

$$\left(sin_{\alpha}(x^{\alpha}) - cos_{\alpha}(x^{\alpha})\right)^{(\alpha)} = cos_{\alpha}(x^{\alpha}) + sin_{\alpha}(x^{\alpha}), \quad (15)$$

$$\left(sin_{\alpha}(x^{\alpha}) \cdot cos_{\alpha}(x^{\alpha})\right)^{(\alpha)} = [cos_{\alpha}(x^{\alpha})]^2 - [sin_{\alpha}(x^{\alpha})]^2, \quad (16)$$

$$\left(\frac{sin_\alpha(x^\alpha)}{cos_\alpha(x^\alpha)}\right)^{(\alpha)} = \frac{[cos_\alpha(x^\alpha)]^2+[sin_\alpha(x^\alpha)]^2}{[cos_\alpha(x^\alpha)]^2}. \quad (17)$$

We note that

$$[cos_\alpha(x^\alpha)]^2 + [sin_\alpha(x^\alpha)]^2 \neq 1, \quad (18)$$

and

$$[cos_\alpha(x^\alpha)]^2 - [sin_\alpha(x^\alpha)]^2 \neq cos_\alpha(2x^\alpha) \quad (19)$$

for $0 < \alpha < 1$.

Remark 2.10: The followings are some local α-fractional differentials.

$$d_\alpha f = f^{(\alpha)} dx^\alpha. \quad (20)$$

$$d_\alpha(f+g) = d_\alpha f + d_\alpha g. \quad (21)$$

$$d_\alpha(f-g) = d_\alpha f - d_\alpha g. \quad (22)$$

$$d_\alpha(\lambda f) = \lambda d_\alpha f, \quad (23)$$

where λ is a constant.

$$d_\alpha(fg) = d_\alpha f \cdot g + f \cdot d_\alpha g. \quad (24)$$

$$d_\alpha\left(\frac{f}{g}\right) = \frac{d_\alpha f \cdot g - f \cdot d_\alpha g}{g^2}. \quad (25)$$

Proposition 2.11 (chain rule for local fractional derivative): *If f is α-fractional differentiable at $u(x_0)$, and u is differentiable at x_0, then $f(u(x))$ is α-fractional differentiable at x_0 and*

$$(f \circ u)^{(\alpha)}(x_0) = f^{(\alpha)}(u(x_0)) \cdot \left(\frac{du}{dx}(x_0)\right)^\alpha. \quad (26)$$

Proof $(f \circ u)^{(\alpha)}(x_0)$

$= \lim_{x \to x_0} \frac{f(u(x)) - f(u(x_0))}{(x-x_0)^\alpha}$

$= \lim_{x \to x_0} \frac{f(u(x)) - f(u(x_0))}{(u(x)-u(x_0))^\alpha} \cdot \frac{(u(x)-u(x_0))^\alpha}{(x-x_0)^\alpha}$

$= \lim_{x \to x_0} \frac{f(u(x)) - f(u(x_0))}{(u(x)-u(x_0))^\alpha} \cdot \lim_{x \to x_0} \left(\frac{u(x)-u(x_0)}{x-x_0}\right)^\alpha$

$= f^{(\alpha)}(u(x_0)) \cdot \left(\frac{du}{dx}(x_0)\right)^\alpha.$ Q.e.d.

III. JUMARIE TYPE OF RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE

In the following, we first give the definition of Jumarie type of fractional derivative.

Definition 3.1: Let α be a real number and p be a positive integer. The modified Riemann-Liouville fractional derivative of Jumarie type ([17]) is defined by

$$\left({}_{x_0}D_x^\alpha\right)[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)} \int_{x_0}^x (x-\tau)^{-\alpha-1} f(\tau) d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x_0}^x (x-\tau)^{-\alpha} [f(\tau) - f(a)] \, d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^p}{dx^p} \left({}_{x_0}D_x^{\alpha-p}\right)[f(x)], & \text{if } p \le \alpha < p+1 \end{cases} \quad (27)$$

In addition, if $\left({}_{x_0}D_x^\alpha\right)^n [f(x)] = \left({}_{x_0}D_x^\alpha\right)\left({}_{x_0}D_x^\alpha\right) \cdots \left({}_{x_0}D_x^\alpha\right)[f(x)]$ exists, then $f(x)$ is called n-th order α-fractional differentiable function, and $\left({}_{x_0}D_x^\alpha\right)^n [f(x)]$ is the n-th order α-fractional derivative of $f(x)$. We note that $\left({}_{x_0}D_x^\alpha\right)^n \neq {}_{x_0}D_x^{n\alpha}$ in general. On the other hand, we define the α-fractional integral of $f(x)$, $\left({}_{x_0}I_x^\alpha\right)[f(x)] = \left({}_{x_0}D_x^{-\alpha}\right)[f(x)]$, where $\alpha > 0$. And $f(x)$ is called α-fractional integrable function. We have the following property.

Proposition 3.2 ([18]): *Let α, β, c be real numbers and $\beta \ge \alpha > 0$, then*

$$\left({}_0D_x^\alpha\right)[x^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \quad (28)$$

$${}_0D_x^\alpha[c] = 0 \quad (29)$$

$$\left({}_0D_x^\alpha\right)[E_\alpha(x^\alpha)] = E_\alpha(x^\alpha). \quad (30)$$

$$\left({}_0D_x^\alpha\right)[sin_\alpha(x^\alpha)] = cos_\alpha(x^\alpha). \quad (31)$$

$$\left({}_0D_x^\alpha\right)[cos_\alpha(x^\alpha)] = -sin_\alpha(x^\alpha). \quad (32)$$

IV. FRACTIONAL ANALYTIC FUNCTIONS

Definition 4.1: Assume that x, x_0 and a_n are real numbers, $x_0 \in (a,b)$, and $0 < \alpha < 1$. If the function $f: [a,b] \to R$ can be expressed as a α-fractional power series, that is, $f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n\alpha}$ on some open interval $(x_0 - r, x_0 + r)$, then we say that $f(x)$ is α-fractional analytic at x_0, where r is the radius of convergence about x_0. If $f: [a,b] \to R$ is continuous on closed interval $[a,b]$ and is α-fractional analytic at every point in open interval (a,b), then f is called a α-fractional analytic function on $[a,b]$.

Proposition 4.2: *Assume that* $0 < \alpha < 1$, $(-1)^\alpha = -1$, $a < b$, $p \neq 0$, $c \in (a,b)$, *and let* $f(x) = (x-c)^p$ *defined on* $[a,b]$.

Case 1. If $p \neq \alpha$, then $f(x)$ is not α-fractional analytic at all points in (a, b).
Case 2. If $p = \alpha$, then $f(x)$ is α-fractional analytic only at $x = c$.

Proof If $p > \alpha$. Since $f^{(\alpha)}(x) = 0$ for all $x \in (a, b)$. It follows that for any positive integer n, $f^{(n\alpha)}(x) = 0$ for all $x \in (a, b)$. And hence, $f(x)$ is not α-fractional analytic at all points in (a, b). Since if (x) is α-fractional analytic at some point $\eta \in (a, b)$, then $f(x)$ is a constant function, which is a contradiction.
If $p < \alpha$. We have

$$f^{(\alpha)}(x) = \begin{cases} 0 & \text{if } x \neq c \\ \text{nonexist} & \text{if } x = c. \end{cases} \quad (33)$$

Thus,

$$f^{(n\alpha)}(x) = \begin{cases} 0 & \text{if } x \neq c \\ \text{nonexist} & \text{if } x = c. \end{cases} \quad (34)$$

for all positive integers n and all $x \in (a, b)$.
And hence, $f(x)$ is not α-fractional analytic at all points in (a, b).
If $p = \alpha$. Then

$$f^{(\alpha)}(x) = \begin{cases} 0 & \text{if } x \neq c \\ \Gamma(\alpha+1) & \text{if } x = c. \end{cases} \quad (35)$$

Therefore, $f^{(n\alpha)}(x) = 0$ for all $n \geq 2$ and all $x \in (a, b)$. So, $f(x)$ is α-fractional analytic only at $x = c$.
Q.e.d.

Theorem 4.3: *Suppose that $0 < \alpha < 1$ and $(-1)^{\alpha} = -1$. If $f: [a, b] \to R$ is α-fractional analytic at $x_0 \in (a, b)$, then*

$$f^{(\alpha)}(x_0) = \left({}_{x_0}D_t^{\alpha}\right)[f(t)](x_0). \quad (36)$$

Proof By Theorem 2.9 in [19], we can immediately obtain this result. Q.e.d.

Corollary 4.4: *Let $0 < \alpha < 1$ and $(-1)^{\alpha} = -1$. If $f: [a, b] \to R$ is α-fractional analytic on $[a, b]$, then*

$$f^{(n)}(x) = \left({}_{x}D_t^{\alpha}\right)^n [f(t)](x) \quad (37)$$

for all positive integers n and all points $x \in (a, b)$.

Proof Using Theorem 4.3 and by induction, we can easily obtain the result. Q.e.d.

Theorem 4.5: *Let $0 < \alpha < 1$. If $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n\alpha)}(x_0)}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}. \quad (38)$$

Proof By Theorem 2.11 in [20] and the above Corollary 4.4, the desired result holds. Q.e.d.

Theorem 4.6 (mean value theorem for local fractional derivative): *Suppose that $0 < \alpha < 1$ and $(-1)^{\alpha} = -1$. If f is α-fractional analytic on $[a, b]$, then there exists $\xi \in (a, b)$ such that*

$$f(b) - f(a) = \frac{f^{(\alpha)}(\xi)}{\Gamma(\alpha+1)} (b - a)^{\alpha}. \quad (39)$$

Proof Using Theorem 2.12 in [19] and the above Theorem 4.3, we obtain the desired result. Q.e.d.

Theorem 4.7: *Let $0 < \alpha < 1$ and $(-1)^{\alpha} = -1$. If f is α-fractional analytic on $[a, b]$ and $f^{(\alpha)}(x) = 0$ for all $x \in (a, b)$, then f is a constant function on $[a, b]$.*

Proof Using mean value theorem for local fractional derivative yields $f(x) = f(a)$ for all $x \in [a, b]$. And hence, the desired result holds. Q.e.d.

V. FRACTIONAL RIEMANN INTEGRAL

Definition 5.1: Let $0 < \alpha < 1$, and $f: [a, b] \to R$. If

$$\lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})^{\alpha}$$

exists, where the partitions of the interval $[a, b]$ are denoted by $[x_{k-1}, x_k]$, $k = 1, \cdots, n$, $x_0 = a, x_n = b$, $\xi_k \in [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, and $\|\Delta\| = \max_{k=1,\cdots,n} \{\Delta x_k\}$. Then we say f is a α-fractional Riemann integrable function on $[a, b]$. And we denoted as

$$\lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})^{\alpha} = \int_a^b f(x)\, dx^{\alpha}, \quad (40)$$

which is called the α-fractional Riemann integral (or local α-fractional integral) of f on $[a, b]$.

Proposition 5.2: *Assume that $0 < \alpha < 1$ and λ is a real number. If $f, g: [a, b] \to R$ are α-fractional Riemann integrable functions on $[a, b]$, then*

(i) $\int_a^b [f(x) + g(x)]\, dx^{\alpha} = \int_a^b f(x) dx^{\alpha} + \int_a^b g(x) dx^{\alpha}$. (41)

(ii) $\int_a^b [f(x)-g(x)]\,dx^\alpha = \int_a^b f(x)dx^\alpha - \int_a^b g(x)dx^\alpha$. (42)

(iii) $\int_a^b \lambda f(x)dx^\alpha = \lambda \int_a^b f(x)dx^\alpha$. (43)

(iv) *If* $c \in (a,b)$, *then*

$$\int_a^c f(x)dx^\alpha + \int_c^b f(x)dx^\alpha = \int_a^b f(x)dx^\alpha. \quad (44)$$

(v) *If* $f \geq 0$, *then* $\int_a^b f(x)dx^\alpha \geq 0$. (45)

(vi) *If* $f \leq g$, *then* $\int_a^b f(x)dx^\alpha \leq \int_a^b g(x)dx^\alpha$. (46)

Proposition 5.3: *If* f *is* α *-fractional Riemann integrable on* $[a,b]$, *then* f *is bounded on* $[a,b]$.

Proof If f is not bounded on $[a,b]$, then for any partition P of $[a,b]$, the function f is not bounded on some interval $[x_{k-1},x_k]$ of P. By choosing the point $\xi_k \in [x_{k-1},x_k]$ in different ways, we can make the quantity $|f(\xi_k)(x_k-x_{k-1})^\alpha|$ as large as desired. Thus, the α -fractional Riemann sum $\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})^\alpha$ can also be made as large as desired in absolute value by changing only the point ξ_k in this interval. That is, f is not α -fractional Riemann integrable on $[a,b]$.

Q.e.d.

Theorem 5.4 (fundamental theorem of local fractional calculus): *Let* $0<\alpha<1$, $(-1)^\alpha=-1$. *If* $F:[a,b]\to R$ *satisfies* $F^{(\alpha)}(x)=f(x)$ *for all* $x\in(a,b)$, *then*

$$\int_a^b f(x)\,dx^\alpha = \Gamma(\alpha+1)\big(F(b)-F(a)\big). \quad (47)$$

Proof Let $\{[x_{k-1},x_k]\}_{k=1,\ldots,n}$ be any partition of the interval $[a,b]$, then by mean value theorem for local fractional derivative and the definition of fractional Riemann integral, we obtain

$$\begin{aligned}
&\Gamma(\alpha+1)\big(F(b)-F(a)\big)\\
&=\Gamma(\alpha+1)\big[\big(F(x_0)-F(x_1)\big)+\big(F(x_1)-F(x_2)\big)+\cdots\\
&\quad+\big(F(x_{n-1})-F(x_n)\big)\\
&=\Gamma(\alpha+1)\Big[\Big(\tfrac{F(x_0)-F(x_1)}{(x_0-x_1)^\alpha}\cdot(x_0-x_1)^\alpha\Big)\\
&\quad+\Big(\tfrac{F(x_1)-F(x_2)}{(x_1-x_2)^\alpha}\cdot(x_1-x_2)^\alpha\Big)+\cdots\\
&\quad+\Big(\tfrac{F(x_{n-1})-F(x_n)}{(x_{n-1}-x_n)^\alpha}\cdot(x_{n-1}-x_n)^\alpha\Big)\Big]\\
&=\big[\big(f^{(\alpha)}(\xi_1)\cdot(x_0-x_1)^\alpha\big)+\big(f^{(\alpha)}(\xi_2)\cdot(x_1-x_2)^\alpha\big)\\
&\quad+\cdots+\big(f^{(\alpha)}(\xi_n)\cdot(x_{n-1}-x_n)^\alpha\big)\big]\\
&=\int_a^b f(x)\,dx^\alpha.
\end{aligned}$$

Q.e.d.

Remark 5.5: If $f:[a,b]\to R$ is a α-fractional analytic function on $[a,b]$, then by fundamental theorem of local fractional calculus, we have

$$\frac{d}{dx^\alpha}\int_a^x f(t)\,dt^\alpha = \Gamma(\alpha+1)f(x) \quad (48)$$

for all $x\in(a,b)$.

On the other hand, if $F:[a,b]\to R$ is α -fractional analytic on $[a,b]$, then

$$\int_a^b F^{(\alpha)}(x)\,dx^\alpha = \Gamma(\alpha+1)\big(F(b)-F(a)\big). \quad (49)$$

Remark 5.6: Fundamental theorem of local fractional calculus cannot hold for non-fractional analytic functions. For example, by Proposition 4.2, we know that $F(x)=x^2$ defined on $[0,1]$, is not $\frac{1}{2}$-fractional analytic function at all points in $[0,1]$. And by the definition of local fractional derivative, we know that $F^{(1/2)}(x)=0$ for all points $x\in[0,1]$. Therefore,

$$\int_0^1 F^{(1/2)}(x)\,dx^{1/2} \neq \Gamma\left(\tfrac{3}{2}\right)\cdot\big(F(1)-F(0)\big). \quad (50)$$

Theorem 5.7: *Suppose that* f *is a* α*-fractional analytic function, and* $u(x), w(x)$ *are differentiable functions, then*

$$\frac{d}{dx^\alpha}\int_{w(x)}^{u(x)} f(t)dt^\alpha = f\big(u(x)\big)\cdot\left(\frac{du}{dx}\right)^\alpha - f\big(w(x)\big)\cdot\left(\frac{dw}{dx}\right)^\alpha. \quad (51)$$

Proof Taking a point c contained in the domain of f. Let $F(x)=\int_a^x f(t)\,dt^\alpha$, then

$$F\big(u(x)\big)=\int_a^{u(x)} f(t)\,dt^\alpha, \quad (52)$$

and

$$F\big(w(x)\big)=\int_a^{w(x)} f(t)\,dt^\alpha. \quad (53)$$

Thus,

$$\int_{w(x)}^{u(x)} f(t)dt^\alpha = F\big(u(x)\big)-F\big(w(x)\big). \quad (54)$$

Therefore, using chain rule for local fractional derivative yields

$$\frac{d}{dx^\alpha}\int_{w(x)}^{u(x)} f(t)dt^\alpha$$

$$= \frac{d}{dx^\alpha} F(u(x)) - \frac{d}{dx^\alpha} F(w(x))$$

$$= F^{(\alpha)}(u(x)) \cdot \left(\frac{du}{dx}(x)\right)^\alpha - F^{(\alpha)}(w(x)) \cdot \left(\frac{dw}{dx}(x)\right)^\alpha$$

$$= f(u(x)) \cdot \left(\frac{du}{dx}(x)\right)^\alpha - f(w(x)) \cdot \left(\frac{dw}{dx}(x)\right)^\alpha .$$

Q.e.d.

Theorem 5.8 (change of variable for local fractional integral): *Suppose that* $0 < \alpha < 1$, $(-1)^\alpha = -1$, $f: [a, b] \to R$ *is a* α*-fractional analytic function, and* $u: [c, d] \to [a, b]$ *is a differentiable function with* $u(c) = a$, $u(d) = b$. *If* $f(u(x)) \left(\frac{du}{dx}\right)^\alpha$ *is* α*-fractional analytic on* $[c, d]$, *then*

$$\int_c^d f(u(x)) \left(\frac{du}{dx}\right)^\alpha dx^\alpha = \int_a^b f(u)\, du^\alpha. \quad (55)$$

Proof Fix $x_0 \in [a, b]$ and let $F(u) = \int_{x_0}^u f(t) dt^\alpha$. Since f is α-fractional analytic on $[a, b]$, it follows from fundamental theorem of local fractional calculus that $F^{(\alpha)}(u) = f(u)$ for all $u \in [a, b]$. Let $g = F \circ u$. Since $f(u(x)) \left(\frac{du}{dx}\right)^\alpha$ is α-fractional analytic on $[c, d]$, using chain rule for local fractional derivative yields

$$g^{(\alpha)}(x) = F^{(\alpha)}(u(x)) \left(\frac{du}{dx}\right)^\alpha = f(u(x)) \left(\frac{du}{dx}\right)^\alpha. \quad (56)$$

And hence g is analytic on $[c, d]$. Furthermore,

$$\int_c^d f(u(x)) \left(\frac{du}{dx}\right)^\alpha dx^\alpha$$

$$= \int_c^d g^{(\alpha)}(x)\, dx^\alpha$$

$$= g(d) - g(c)$$

$$= F(u(d)) - F(u(c))$$

$$= \int_{x_0}^{u(d)} f(t)\, dt^\alpha - \int_{x_0}^{u(c)} f(t)\, dt^\alpha$$

$$= \int_{u(c)}^{u(d)} f(t)\, dt^\alpha$$

$$= \int_a^b f(u)\, du^\alpha.$$

Q.e.d.

Theorem 5.9 (integration by parts for local fractional calculus): *If* f, g *are* α*-fractional analytic functions on* $[a, b]$, *then*

$$\int_a^b f(x) \cdot g^{(\alpha)}(x)\, dx^\alpha$$
$$= f(x) \cdot g(x)|_a^b - \int_a^b g(x) \cdot f^{(\alpha)}(x)\, dx^\alpha. \quad (57)$$

Proof Since by product rule for local fractional derivative, we have

$$f(x) \cdot g^{(\alpha)}(x) = (f \cdot g)^{(\alpha)}(x) - f^{(\alpha)}(x) \cdot g(x). \quad (58)$$

It follows from fundamental theorem of local fractional calculus that

$$\int_a^b f(x) \cdot g^{(\alpha)}(x)\, dx^\alpha$$
$$= \int_a^b (f \cdot g)^{(\alpha)}(x)\, dx^\alpha - \int_a^b f^{(\alpha)}(x) \cdot g(x)\, dx^\alpha$$
$$= f(x) \cdot g(x)|_a^b - \int_a^b g(x) \cdot f^{(\alpha)}(x)\, dx^\alpha .$$

Q.e.d.

The followings are some indefinite local fractional integral formulas.

Proposition 5.10: *Let* $0 < \alpha < 1$ *and* C *be a constant. Then*

$$\int E_\alpha(x^\alpha) dx^\alpha = \Gamma(\alpha + 1) \cdot E_\alpha(x^\alpha) + C, \quad (59)$$

$$\int sin_\alpha(x^\alpha) dx^\alpha = -\Gamma(\alpha + 1) \cdot cos_\alpha(x^\alpha) + C, \quad (60)$$

$$\int cos_\alpha(x^\alpha) dx^\alpha = \Gamma(\alpha + 1) \cdot sin_\alpha(x^\alpha) + C, \quad (61)$$

$$\int \sum_{k=0}^{\infty} \frac{a_n}{\Gamma(k\alpha+1)} x^{k\alpha}\, dx^\alpha$$
$$= \Gamma(\alpha + 1) \cdot \sum_{k=0}^{\infty} \frac{a_n}{\Gamma((k+1)\alpha+1)} x^{(k+1)\alpha} + C, \quad (62)$$

where $\sum_{k=0}^{\infty} \frac{a_n}{\Gamma(k\alpha+1)} x^{k\alpha}$ is a α-fractional analytic function on some closed interval.

Remark 5.11: We note that the formulas

$$\int 1 dx^\alpha = x^\alpha + C, \quad (63)$$

and

$$\int x^{k\alpha} dx^\alpha = \frac{\Gamma(k\alpha+1)\cdot\Gamma(\alpha+1)}{\Gamma((k+1)\alpha+1)} \cdot x^{(k+1)\alpha} + C \quad (64)$$

are false (k is any positive integer). Since by Proposition 4.2, the local α-fractional derivatives of the functions $F(x) = x^\alpha + C$ and $G(x) = \frac{\Gamma(k\alpha+1)\cdot\Gamma(\alpha+1)}{\Gamma((k+1)\alpha+1)} \cdot x^{(k+1)\alpha} + C$, are $F^{(\alpha)}(x) = 0$ and $G^{(\alpha)}(x) = 0$ for all $x \neq 0$.

VI. CONCLUSION

From the above discussion, we know that the fractional analytic function plays an important role in local fractional calculus. And the Jumarie type of fractional calculus is equivalent to local fractional calculus when the studied function is fractional analytic. As well, we discussed some important properties in local fractional calculus, for example, fractional mean value theorem, fundamental theorem of local fractional calculus, fractional integration by parts, fractional change of variable and so on. In fact, these theorems are natural generalizations of the ones in classical calculus. In the future, the results in local fractional calculus we obtained will be used to extend the research topics to applied science and engineering mathematics.

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