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Analysis of Laplace Transform & Its Specific Applications in Engineering

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ABSTRACT

Entering in the late 20th century after being popularized by a famous Electrical Engineer, knowledge on how to do the Laplace Transform has become a necessity for many fields. Anyone having background of Mathematics including Engineering & Sciences are always exposed to Differential Equations, Laplace transform etc. Through this paper, we have tried to bring out some specific Applications of Laplace Transform in different Engineering fields specially solving Differential equations by using Laplace Transform in turn showcasing more important uses of the transform.

Keywords : Laplace Transform, Differential Equation, Inverse Laplace Transform.

I. INTRODUCTION

Laplace transform, in Mathematics, a particular integral transform invented by the French mathematician Pierre-Simon Laplace (1749–1827), and systematically developed by the British physicist Oliver Heaviside (1850–1925), to simplify the solution of many Differential Equations that describe physical processes. A Laplace transform is an extremely diverse function that can transform a real function of time t to one in the complex plane s, referred to as the frequency domain.

Laplace transform makes it easier to solve the increasing complexity of engineering problems and has wide applications of Laplace in Electric circuit analysis, Communication Engineering, Nuclear Physics as well as Automation Engineering, Control Engineering and Signal processing, Probability theory, determination of transfer function in Mechanical system and many more. The ready tables of Laplace Transforms reduce the problems of solving differential equations to mere algebraic manipulation.

II. DEFINITION OF INTEGRAL TRANSFORM

Let *K(s, t)* be a function of two variables *s* and *t* where *s* is a parameter [$s \in$ R or C] independent of t. Then the function *F(s)* defined by an Integral which is convergent. i.e.,

$F(s) = \int_{-\infty}^{\infty} K(s,t)\ f(t)\ dt$ is called the Integral Transform of the function f(t) and is denoted by I{f(t)} where K(s, t) is kernel of the transformation.

A. Definition of Laplace Transform

If kernel *K(s, t)* is defined as $K(s,t) = \begin{cases} 0, & \forall\ t < 0\ \ or\ \frac{1}{t} > 0 \\ e^{-st} & \forall\ t \geq 0 \end{cases}$ then

$F(s) = \int_{0}^{\infty} e^{-st}\ f(t)\ dt$ is called "Laplace Transform" of the function f(t) and is also denoted by L{f(t)} or $\overline{f}(s)$.

$$\therefore L[f(t)]\ or\ \overline{f}(s)\ \ or\ F(s) = \int_{0}^{\infty} e^{-st}\ f(t)\ dt$$

B. Inverse Laplace Transform

If L {f (t) } = $\overline{f}(s) \Rightarrow$ f (t) = L^{-1} { $\overline{f}(s)$ }. Then f(t) is called the inverse Laplace Transform of $\overline{f}(s)$. Here operator L which transforms f(t) into $\overline{f}(s)$ is called "The Laplace Transforms Operator".

C. Piecewise or sectionally continuous

A function f (t) is said to be piecewise or sectionally continuous on a closed interval $a \leq t \leq b$, if it is defined on that interval can be divided into finite number of sub-intervals in each of which f(t) is continuous and has finite left limit and right hand limits.

$\lim_{t \to 0^-} f(t) = \lim_{t \to 0^+} f(t) = \lim_{t \to 0} f(t) = \text{finite f}$ say

$\forall\ a \leq t \leq b$, therefore, f is continuous.

D. Functions of an Exponential order

A function f(t) is said to be an exponential order α as t tends to ∞ if there exist a positive real number M a number α and a finite number t_0 such that $| f(t) | < M e^{\alpha t}$ **or** $| e^{-\alpha t} f(t) | < M, \forall\ t \geq t_0$.

Note: If a function f(t) is of an exponential order α. It is also of β such that $\beta > \alpha$.

III. LAPLACE TRANSFORM OF SOME SPECIAL FUNCTIONS

A. Periodic function

A function f (t) is said to be a periodic function of period T > 0 if

f(t) = f(T + t) = f(2T + t) = f(3T + t) = ... = f(nT + t).

sin t, cos t are periodic functions of period 2π.

The Laplace transform of a piecewise periodic function f (t) with period T is

$$L\{ f(t) \} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} . f(t)\, dt \; ; s > 0$$

B. Unit step function or Heaviside Unit function

In engineering, many times we come across such fractions of which inverse Laplace is either very difficult or cannot be obtained by the known formulae. To overcome such problem, Unit step function (Heaviside's Unit Function) has been introduced. The unit step function is defined as follows

$$u(t-a) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$$

Where a is always positive.

As a particular case when a = 0 then

$$u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$$

Also,

$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a) dt = \int_0^a e^{-st}(0)\, dt + \int_a^\infty e^{-st}(1)\, dt$ $L[u(t-a)] = \left. \frac{e^{-st}}{-s} \right|_a^\infty = \frac{e^{-as}}{s}$ and in particular when $a = 0$, $L[u(t)] = \frac{1}{s}$

Generally the unit step function in mechanical engineering comes into picture as a force suddenly applied to a machine or a machine component, where as in electrical engineering it manifests as an electromotive force of a battery in circuit.

C. Dirac's Delta function or Unit impulse function

A force of very high magnitude applied for an instant producing a large effect is called impulse .The function representing the impulse is called dirac-Delta function.

More precisely, Dirac-Delta function is the limiting form of the function $\delta_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & , a \leq t \leq a+\varepsilon \\ 0 & , \text{otherwise} \end{cases}$ as $\varepsilon \to 0$.This fact can be further represented as $\delta(t-a) = \begin{cases} \infty & , t = a \\ 0 & , t \neq a \end{cases}$ such that $\int_0^\infty \delta(t-a) dt = 1 \; for \; a \geq 0$

The Dirac-Delta function is mainly originated from the concept of strongly peaked or concentrated functions. In electrical circuits strongly peaked currents of extremely short duration occurs frequently in switching processes. In mechanics, the impulse of the blow is equal to momentum when a body is set in motion from rest by a sudden -blow.

$$L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)dt$$

$$= \int_0^a e^{-st}(0)\,dt + \int_a^{a+\varepsilon} e^{-st}(\frac{1}{\varepsilon})\,dt + \int_{a+\varepsilon}^\infty e^{-st}(0)\,dt$$

$$= \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = \frac{e^{-as}}{s}\left[\frac{1-e^{-s\varepsilon}}{\varepsilon}\right]$$

Taking limit as $\varepsilon \to 0$

$$L[\delta(t-a)] = \lim_{\varepsilon\to 0}\frac{e^{-as}}{s}\left[\frac{1-e^{-s\varepsilon}}{\varepsilon}\right] = e^{-as}$$

IV. APPLICATIONS

A. In Frequency Response system

An important property of Laplace Transform systems is that if the input given to the system is sinusoidal, then the output must be sinusoidal at the same frequency but with different magnitude and phase. These differences as a function of frequency are known as the frequency response of the system. Frequency-domain methods are often used for analyzing LTI single-input/single output (SISO) systems.

B. Digital Signal Processing

It's use is observed while sending signals over two-way communication medium (for ex. FM/AM stereos, 2 way radio sets, cellular phones.) where medium wave's time functions are converted to frequency functions.

C. Traffic Engineering

It is widely implemented for quantifying speed control through modelling of road bumps by converting to hollow rectangular shape. In doing so, the vehicle is considered as the classical one-degree-of-freedom system whose base follows the road profile, thus approximated by Laplace Transform.

D. Medical Field

One can corelate blood-velocity/time wave form over cardiac cycle from common femoral artery as an approximation of Laplace Transform.

E. Electric circuits

Ex. Use Laplace transform method to obtain the charge at any instant of a capacitor which is discharged in R–C–L circuit, after the switch is closed if R = 2.25 ohms, self-inductance L = 1 Henry, capacitance, C = 2 farads, and the capacitor has initially a charge of 100 coulombs. Initially the switch is open and therefore, no current is flowing.

Solution: The L-C-R circuit equation is

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{E}{L}$$

On Substituting values, we get

$$\frac{d^2q}{dt^2} + \frac{9}{4}\frac{dq}{dt} + \frac{q}{2} = 0$$

Taking Laplace transform on both the side, we get

$$\left[s^2 + \frac{9}{4}s + \frac{1}{2}\right]\bar{q}(s) - 100s - 225 = 0$$

$$\bar{q}(s) = \frac{100(4s+9)}{(4s^2+9s+2)} = \frac{100}{7}\left[\frac{32}{4s+1} - \frac{1}{s+2}\right]$$

$$\bar{q}(s) = \frac{100}{7}\left[\frac{8}{s+\frac{1}{4}} - \frac{1}{s+2}\right]$$

Taking inverse Laplace transform on both sides, we get $q(t) = \frac{100}{7}\left[8e^{-\frac{t}{4}} - e^{-2t}\right]$

F. Vibration Mechanics

Ex. Obtain the equation for the forced oscillations of mass *m* attached to the lower end of an elastic spring whose upper end is fixed and whose stiffness is *k*, when the driving force is F_0 *sinat*. Solve this equation (Using the Laplace Transforms) when $a^2 \neq k/m$, given that initially velocity and displacement (from equilibrium position) are zero.

Solution: This problem deals with forced oscillations (without damping)

Let the periodic force be F_0 *sinat* then equation of motion becomes

$$m\frac{d^2x}{dt^2} = mg - k\,(e + x) + F_0\, sinat$$

where, x is length of stretched portion of the spring (displacement) after time t,

e is the elongation produced by the mass m,

k is the restoring force per unit stretch of the spring due to elasticity;

a is any arbitrary constant and p is any scalar.

Since tension $mg = k\,e$, equation becomes

$$\frac{d^2x}{dt^2} + n^2x = \frac{F_0}{m}\, sinat$$

Taking L.T. on both sides and using $x(0) = 0, x'(0) = 0$

$$\overline{x(s)} = \frac{a\,F_0}{m}\,\frac{1}{(s^2+a^2)(s^2+n^2)} \quad \mu = \sqrt{\frac{k}{m}}$$

Resolving by Partial fractions we have A = 0, B = $\frac{1}{n^2-a^2}$, C = 0, D = $\frac{-1}{n^2-a^2}$

Thus, we get $\overline{x(s)} = \frac{a\,F_0}{m\,(n^2-a^2)}\left[\frac{1}{(s^2+a^2)} - \frac{1}{(s^2+n^2)}\right]$

Taking inverse Laplace Transform

$x(t) = \frac{F_0}{m\,n\,(n^2-a^2)}\,[n\, sinat - a\, sinnt]$, where $n = \sqrt{\frac{k}{m}}, \frac{k}{m} \neq a^2$ is the required solution.

V. CONCLUSIONS

Thus, from the above analysis of definition & its forms as well as illustrious various applications of Laplace transform one can infer that it is an indispensable tool in solving linear ordinary and partial differential equations with constant coefficients under suitable initial and boundary conditions by first finding the general solution and then evaluating from it the arbitrary constants. Modern society would have witnessed set back in rapid increase of technology and growth might have stopped if such varied and diverse applications are not brought forward. For partial differential equations involving two independent variables, Laplace transform is applied to one of the variables and the resulting differential equation in the second variable is solved by the usual method of ordinary differential equations. Hopefully, the applications discussed of Laplace transform in this paper will prove beneficial to students and researchers.

VI.REFERENCES

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