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Some Generalised Connections in Finsler Space

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ABSTRACT

In the present paper we discuss some generalized Cartan connection $GC\Gamma$ and Berwald connection with torsion $B\Gamma T$ which are defined as a Finsler connection satisfying the properties of Cartan connection $C\Gamma$ and Berwald connection $B\Gamma$ respectively. Both $GC\Gamma$ and $B\Gamma T$ exists for (h) h-torsion T^i_{jk} and (v) hv-torsion P^i_{jk}. We have also found few results in the case of Berwald connection with torsion $B\Gamma T$.

Keywords : Finsler Connection, Cartan Connection, Berwald Connection.

I. INTRODUCTION

One can define many different kind of connections in Finsler space. In this work we are focusing two such connections in Finsler space, namely Carton $C\Gamma$ and Berwald $B\Gamma$ connection. Many generalizations of these connection are possible. Wagner [1], Hashiguchi-Ichijyo [2] and Hashiguchi [3] have introduced the notion of generalized Cartan connection $GC\Gamma$ and defined a generalized Berwald space with respect to this $GC\Gamma$. Matsumoto [4] has also generalized Berwald space and defined the notion of Berwald connection with torsion $B\Gamma T$. These generalizations are in context of (h) h-torsion T^i_{jk}. Matsumoto [5] while discussing hypersurface of a Finsler space obtained an interesting Berwald connection in context of (v) hv-torsion P^i_{jk}. M. Kitayama and S. Kikuchi [6] have introduced on metrical Finsler connections with torsion T^i_{jk} of generalized Finsler spaces. In this work we have discussed these connections and have found some new interesting results.

Let F^n = (M, L) be an n-dimensional Finsler space where $L(x,y)$ is the fundamental function. The fundamental tensor g_{ij} is given by $g_{ij} = (\dot{\partial}_i\dot{\partial}_j L^2)/2$, where $\dot{\partial}_i$ denotes the partial differentiation by y^i. We will denote a Finsler connection Γ as $F\Gamma = (F^i_{jk}, N^i_k, C^i_{jk})$.

The Carton connection $C\Gamma$ is defined as following system of axioms [8]:

(1.1) (a) $g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - N^r_k \frac{\partial g_{ij}}{\partial y^r} - g_{rj}F^r_{ik} - g_{ir}F^r_{jk} = 0$,

(b) $D^i_k \equiv F^i_{0k} - N^i_k = 0$,

(c) $T^i_{jk} = F^i_{jk} - F^i_{kj} = 0$, (d) $S^i_{jk} = C^i_{jk} - C^i_{kj} = 0$,

(e) $g_{ij}|_k = \frac{\partial g_{ij}}{\partial y^r} - g_{rj}C^r_{ik} - g_{ir}C^r_{jk} = 0$.

where subscript 0 the contraction for means y^j and $C^i_{jk} = \frac{1}{2}g^{ir}\left(\frac{\partial g_{kr}}{\partial y^j} + \frac{\partial g_{jr}}{\partial y^k} - \frac{\partial g_{jk}}{\partial y^r}\right)$.

The Berwald connection $B\Gamma$ are defined by the following systems of axioms [7]:

(1.2) (a) $L_{|k} = 0$, (b) $D^i_k = 0$,

(c) $T^i_{jk} = 0$,

(d) $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj} = 0$, (e) $C^i_{jk} = 0$.

We now consider the generalized Cartan connection $GC\Gamma$. Which is defined as a Finsler connection satisfying the axioms of $C\Gamma$ except (1.1c). Also a generalized Berwald connection $B\Gamma T$ is defined as a Finsler connection satisfying the axioms of $B\Gamma$ except (1.2c). Both $GC\,\Gamma$ and $B\Gamma T$ are with exists (h) h-torsion T^i_{jk} and (v) hv-torsion P^i_{jk} [4].

For a Finsler connection $(F^i_{jk}, N^i_k, C^i_{jk})$, the equations (1.2a), (1.2b), (1.2e) can be expressed in terms of its torsions T^i_{jk}, P^i_{jk} as follows: [9]

(1.3) $N^i_k = G^i_k - \left(\left(T^i_{k0} + P^i_{0k}\right) + \dot\partial_k\left(T^i_{00} - P^i_{00}\right)\right)/2$,

(1.4) $F^i_{jk} = \dot\partial_j N^i_k - P^i_{kj}$,

(1.5) $C^i_{jk} = 0$,

where G^i_k is the non-linear connection of $B\Gamma$.

When P^i_{jk}= 0, one gets the coefficients of a Berwald connection with torsion $B\Gamma T$ of Matsumoto [4]. According to Hashiguchi [10], a generalized Cartan connection $GC\Gamma$ with torsion T^i_{jk} is uniquely determined for an arbitrarily given tensor T^i_{jk}. Following Matsumoto [4], in the case of $B\Gamma T$, the torsion T^i_{jk} satisfy the condition

(1.6) $y^r\left(\dot\partial_k T^i_{jr} - \dot\partial_j T^i_{kr}\right) = 0$.

In order to generalize the connections the torsions T^i_{jk}, P^i_{jk} should satisfy some conditions. In our work we will use following definitions.

DEFINITION (1):

A Finsler connection $FN\Gamma$ is said to be N-connection of a Finsler connection $F\Gamma$, where $FN\Gamma = \left(\dot\partial_j N^i_k, N^i_k, 0\right)$, $\dot\partial_j N^i_k = F^i_{jk} + P^i_{kj}$ and the (h)h-torsion Q^i_{jk} of $FN\Gamma$ is exressed by the torsions T^i_{jk} , P^i_{jk} of $F\Gamma$ as

(1.7) $Q^i_{jk} = T^i_{jk} - \left(P^i_{jk} - P^i_{kj}\right)$.

Where $Q^i_{jk} = \dot\partial_j N^i_k - \dot\partial_k N^i_j$

We now propose following propositions:

PROPOSITION (1): Let $F\Gamma$ be a Finsler connection. If N^i_k of $F\Gamma$ is (1) p-homogeneous, the tensor Q^i_{jk} given by (1.7) satisfies the condition

(1.8) $y^r\left(\dot\partial_k Q^i_{jr} - \dot\partial_j Q^i_{kr}\right) = 0$.

In the case of $P^i_{jk} = 0$, the condition (1.8) is the $B\Gamma$-condition (1.5).

In the case of $GC\Gamma$ we can choose T^i_{jk} arbitrarily, but the condition (1.8) is implicitly imposed on T^i_{jk} together with P^i_{jk}. In the case of $B\Gamma T$ the condition (1.8) is explicitly imposed on T^i_{jk} as the $B\Gamma$ –condition, since we assume P^i_{jk}=0 for $B\Gamma T$. This is the reason for the difference between the arbitrariness of T^i_{jk} in $GC\Gamma$ and the one in $B\Gamma T$. One should note (following Matsumoto [4]) that a (0) p-homogeneous alternate tensor Q^i_{jk} satisfies (1.8) and can be put in the following form

(1.9) $Q^i_{jk} = (\dot\partial_k A^i_j - \dot\partial_j A^i_k)/2$.

where A^i_j is an arbitrary (1) p-homogeneous tensor. Q^i_{jk} can also be written as

(1.10) $Q^i_{jk} = A^i_{jk} + y^r(\dot\partial_k A^i_{jr} - \dot\partial_j A^i_{kr})/2$,

where A^i_{jk} is an arbitrary (0) p-homogeneous alternate tensor. So we now have

PROPOSITION (2): For a given (0) p-homogeneous alternate tensor T^i_{jk}, a tensor P^i_{jk} satisfying (1.8) is expressed in the form

(1.11) $P^i_{jk} = (T^i_{jk} - Q^i_{jk})/2 + B^i_{jk}$,

where Q^i_{jk} is a tensor given by (1.9) or (1.10), and B^i_{jk} is an arbitrary (0) p-homogeneous symmetric tensor. For a given (0) p-homogeneous tensor P^i_{jk} , an alternate tensor T^i_{jk} satisfying (1.8) is expressed in the form

(1.12) $T^i_{jk} = P^i_{jk} - P^i_{kj} + Q^i_{jk}$,

where Q^i_{jk} is a tensor given by (1.9) or (1.10).

If we assume the p-homogeneity of N^i_k for a Finsler connection, we have $P^i_{j0} = -D^i_k$, since $P^i_{jk} = \dot\partial_k N^i_j - F^i_{kj}$. It is well known (Matsumoto [4]) that if N^i_k of $F\Gamma$ is (1) p homogeneous, the deflection tensor D^i_k vanishes, i.e. $D^i_k = 0$, if and only if

(1.13) $P^i_{j0} = 0$.

II. GENERALIZED BERWALD CONNECTIONS IN FINSLER SPACE:

Now we consider Finsler connections which are p-homogeneous. In such case the torsions T^i_{jk}, P^i_{jk} should satisfy the conditions (1.8) and (1.13). In other words for a (0) p-homogeneous tensors T^i_{jk}, P^i_{jk} , there exists a unique Finsler connection $F\Gamma$ satisfying (1.2a), (1.2b), (1.2e) whose (h)h-and (v)hv-torsion tensors are the given T^i_{jk} , P^i_{jk} respectively. If T^i_{jk} and P^i_{jk} satisfy the conditions (1.8), (1.13), where Q^i_{jk} is tensor given by (1.7) then coefficients of $F\Gamma$ are given by (1.3), N^i_k of $F\Gamma$ is also expressed in the form [9]

(2.1) $N_k^i = G_k^i - (Q_{k0}^i + \dot{\partial}_k Q_{00}^i)/2$

DEFINITION (2):

A Finsler connection $GB\Gamma = (F_{jk}^i, N_k^i, G_{jk}^i)$ satisfying (1.2a), (1.2b) and (1.2e) is called a generalized Berwald connection, where $G_{jk}^i = \dot{\partial}_k G_j^i = \dot{\partial}_j \dot{\partial}_k G^i$. So we now have

THEOREM (1):

A Finsler connection $GB\Gamma$ is generalized Berwald connection if it satisfies the Berwald connection $B\Gamma$.

We have already shown that, by taking $GC\Gamma = (F_{jk}^i, N_k^i, C_{jk}^i)$ a generalized Berwald connection $GB\Gamma$ can be obtained. Its (h)h-torsion is an arbitrarily given (0)p-homogeneous alternate tensor T_{jk}^i. It can be easily seen that this $GB\Gamma$ is nothing but the C-zero connection $(F_{jk}^i, N_k^i, 0)$ of $GC\Gamma$. Also, if we take T_{jk}^i=0, the $GC\Gamma$ becomes $C\Gamma = (\Gamma_{jk}^{*i}, G_k^i, g_{jk}^i)$ and the $GB\Gamma$ is the Rund connection $R\Gamma = (\Gamma_{jk}^{*i}, G_k^i, 0)$. Therefore we can state

THEOREM (2):

The C-zero connection of a generalized Cartan connection is a generalized Berwald connection.

In other words, we can state:

THEOREM (3):

The Rund connection is a generalized Berwald connection obtained from the Cartan connection at its C-zero connection.

The above definition of a generalized Berwald connection $GB\Gamma$ has various advantages. We now introduce following definitions for $GC\Gamma$ and $GB\Gamma$:

DEFINITION (3):

A Finsler connection $GC\Gamma = (F_{jk}^i, F_{0k}^i, g_{jk}^i)$ satisfying $g_{ij|k} = 0$ is called a generalized Cartan connection, and a Finsler connection $GC\Gamma = (F_{jk}^i, F_{0k}^i, 0)$ satisfying $L_{|k} = 0$ is called a generalized Berwald connection.

THEOREM (4):

A Finsler connection $GC\Gamma$ is generalized Cartan connection if it satisfies the Carton connection $C\Gamma$.

THEOREM (5):

A Finsler connection $GC\Gamma$ is generalized Berwald connection if it satisfies the Berwald connection $B\Gamma$.

DEFINITION (4):

A Finsler connection is called linear, if the coefficients F_{jk}^i depend on position alone. A Finsler space is a generalzed Berwald space, if we can introduce a linear $B\Gamma T$ (Matsumoto [4]).

Now, if a generalized Berwald connection $GB\Gamma$ is linear, we have $P_{jk}^i = \dot{\partial}_k F_{0j}^i - F_{kj}^i = 0$. So the $GB\Gamma$ becomes a $G\Gamma$T. Consequently we can also define a generalized Berwald space in terms of $GB\Gamma$. So we have our final result:

THEOREM (6):

A Finsler space is a generalized Berwald connection $GB\Gamma$, if there exist a linear $G\Gamma$T.

CONCLUSION:

In this paper we have found some interesting results related to generalized Cartan connection $GC\ \Gamma$. We also introduced a Berwald connection with torsion $B\Gamma$T. Both $GC\ \Gamma$ and $B\Gamma$T have been discussed for the two cases of (h) h-torsion T_{jk}^i and (v) hv-torsion P_{jk}^i. In this paper we have also investigated Finsler connections which are p-homogeneous. An important observation in case of (0) p-homogeneous tensors T_{jk}^i *and* P_{jk}^i , is that there exists a unique Finsler connection $F\Gamma$ satisfying Berwald connection, whose (h)h-and (v)hv-torsion tensors are the given T_{jk}^i , P_{jk}^i respectively. We have found that connection $GB\Gamma$ is generalized Berwald connection if it satisfies the Berwald connection $B\Gamma$. Also the C-zero connection of a generalized Cartan connection is a generalized Berwald connection. In other words, the Rund connection is a generalized Berwald connection obtained from the Cartan connection at its C-zero connection. In investigation we have found that a Finsler connection $GC\Gamma$ is generalized Cartan connection if it satisfies the Carton connection $C\Gamma$

and the Finsler connection $GC\Gamma$ generalized Berwald connection if it satisfies the Berwald connection $B\Gamma$. Also Finsler space is a generalized Berwald connection $GB\Gamma$, if there exist a linear $G\Gamma T$.

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