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Multiplication and Addition of Tetrajection Operator

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ABSTRACT

In this paper we define a tetrajection operator in analogue to a projection and a trijection operator. We study condition for multiplication of two tetrajections to be a tetrajection and relation between their ranges and null spaces. We obtain condition for addition of two tetrajections to be a tetrajection and relation between their ranges and null spaces.

Keywords : Projection, Trijection, Tetrajection

INTRODUCTION

In this paper we have introduced a new type of linear operator called tetrajection operator on a linear space as well as on a Hilbert space. It is a generalization of projection operator as defined in Dunford and Schwartz [2] page 37 and Rudin [3] page 126 in the sense that every projection is a tetrajection but a tetrajection is not necessarily a projection.

A projection operator E on a Hilbert space H, as defined in Simmons G. F., p-274 [4] by the formula

$E^2 = E$ and $E^* = E$, where E^* is the adjoint of E.

A trijection operator has been defined by Chandra [1] by the formula $E^3 = E$. We study condition for multiplication of two tetrajections to be a tetrajection and relation between their ranges and null spaces. We obtain condition for addition of two tetrajections to be a tetrajection and relation between their ranges and null spaces.

A special type of operator:

We define E, a linear operator on a linear space L to be a tetrajection if $E^4 = E$. If E is a projection operator on the linear space L, then $E^2 = E$.

$$\Rightarrow E^3 = E^2.E = E.E = E^2 = E$$

$$\text{and } E^4 = E^3.E = E.E = E^2 = E$$

This shows that a projection is necessarily a tetrajection but a tetrajection is not necessarily a projection or a trijection. This would be clear from the following example:

Let C be the set of the set of all complex numbers and let $L = C^2$. Let $E(x,y) = (x, \omega y)$ where x, y are the complex numbers and ω is an imaginary cube root of unity.

$$E^2 (x,y) = E(E(x,y)) = E(x,\omega y) = (x,\omega^2 y) \neq E(x,y)$$

Hence $E^2 \neq E$

Therefore E is not a projection.

$$E^3 (x,y) = E(E^2(x,y)) = E(x,\omega^2 y) = (x,\omega^3 y) \neq E(x,y)$$

So $E^3 \neq E$.

Hence E is not a trijection.

$E^4(x,y) = E(E^3(x,y)) = E(x,y)$

So $E^4 = E$.

Hence E is a tetrajection but it is neither a projection nor a trijection.

We define null space of the tetrajection E as {z:Ez =0} which we denote by N (or N_E) and range of the tetrajection E is defined by {z:E^3z = z} which we denote by R (or R_E).

A linear operator E defined on a Hilbert space H is called tetrajection if E^4=E and E^* = E, where E^* is the adjoint of E.

Theorem(3.i):

If A and B are the tetrajection on closed linear subspaces M and N of H, then AB is a tetrajection ⇔ AB=BA

Also in this case AB is the tetrajection on $M \cap N$ and $N_{AB} = N_A + N_B$.

Proof:

Let $AB = BA$, then

$(AB)^2 = (AB)(AB) = A(AB)B = A(AB)B$

$= (AA)(BB) = A^2B^2$

$(AB)^3 = (AB)^2AB = A^2B^2AB = A^2B(BA)B$

$= A^2B(AB)B = A^2(BA)B^2$

$= A^2(AB)B^2$

$A^2ABB^2 = A^3B^3$

$Hence\ (AB)^4 = (AB)^3AB = A^3B^3AB$

$= A^3B^2(BA)B$

$= A^3B^2(AB)B = A^3B(BA)B^2$

$= A^3B(AB)B^2 = A^3(BA)B^3$

$= A^3(AB)B^3$

$= A^3ABB^3 = A^4B^4 = AB$

(Since A and B are tetrajections)

Also $(AB)^* = B^*A^* = BA = AB$

Hence AB is a tetrajection.

Conversely, let AB be a tetrajection, the $(AB)^* = AB \Rightarrow B^*A^* = AB$

$\Rightarrow BA = AB$

Now again if AB is a tetrajection, then

$R_{AB} = \{x: (AB)^3x = x\}$

$= \{x: A^3B^3x = x\}$

Let x ϵ $M \cap N$, then x ϵ M and x ϵ N

$\Rightarrow A^3x = x\ and\ B^3x = x$

$Hence\ (AB)^3x = A^3B^3x = A^3x = x$

$\Rightarrow x \epsilon R_{AB}$

Therefore $x \epsilon M \cap N \Rightarrow x \epsilon R_{AB}$

Hence $M \cap N \subseteq R_{AB}$ ---------------------------- (3.11)

Again, $let\ x \epsilon R_{AB}, then\ (AB)^3x = x \Rightarrow A^3B^3x = x$

Therefore $x = A^3(B^3x) = A(A^2B^3x) \epsilon M$

Since $(BA)^3 = B^3A^3, (AB)^3 = A^3B^3\ and$

$BA = AB\ we\ have\ B^3A^3 = A^3B^3$

Hence $x = A^3B^3x = B^3A^3x = B(B^2A^3x) \epsilon M$

Now $x \epsilon M\ and\ x \epsilon N, so\ x \epsilon M \cap N$

Thus $x \epsilon R_{AB} \Rightarrow x \epsilon M \cap N$

Hence $R_{AB} \subseteq M \cap N$ ---------------------------- (3.12)

From (3.11) and (3.12), we get

$R_{AB} = M \cap N = R_A \cap R_B$

Also, $N_{AB} = \{z: (AB)^3 z = 0\} = \{z: (A^3B^3)z = 0\}$

Let $z \epsilon N_{AB}, then\ A^3(B^3z) = 0$

Therefore $B^3z \epsilon N_{A^3} = N_A$ as A is a tetrajection.

[$z \epsilon N_{A^3} \Rightarrow A^3z = 0 \Rightarrow A^4z = A(A^3z) = A(0) = 0$

$\Rightarrow Az = 0$

$\Rightarrow z \epsilon N_A$

Therefore $N_{A^3} \subseteq N_A$

Conversely, $z \epsilon N_A \Rightarrow Az = 0$

$\Rightarrow A^2z = A(Az) = A(0) = 0$

$\Rightarrow A^3z = A(A^2z) = A(0) = 0$

$\Rightarrow$ z ϵ N_{A^3}

Therefore $N_A \subseteq N_{A^3}$

Thus $N_{A^3} \subseteq N_A$ and $N_A \subseteq N_{A^3}$

Hence $N_{A^3} = N_A$]

Since $B(z - B^3z) = Bz - B^4z$

$= Bz - Bz = 0$

So $z - B^3z \epsilon N_B$

Therefore $z = B^3z + (z - B^3z) \epsilon N_A + N_B$

Thus $z \epsilon N_{AB} \Rightarrow z \epsilon N_A + N_B$

Hence $N_{AB} \subseteq N_A + N_B$ ---------------------------- (3.13)

Let $z \epsilon N_A + N_B$, then we can write

$z = z_1 + z_2\ where\ z_1 \in N_A\ and\ z_2 \epsilon N_B$

$\Rightarrow A(z_1) = 0\ and\ B(z_2) = 0$

Now $(AB)z = (AB)(z_1 + z_2)$

$= ABz_1 + ABz_2$

$= B(Az_1) + A(Bz_2)$

$= B(0) + A(0)$

$= 0 + 0 = 0$

Therefore z ϵ N_{AB}

Thus z ϵ $N_A + N_B \Rightarrow$ z ϵ N_{AB}

Hence $N_A + N_B \subseteq N_{AB}$ ---------------------------- (3.14)

From (3.13) and (3.14), we get

$N_{AB} = N_A + N_B$

Theorem (3.II) :

If A and B are tetrajections on a Hilbert Space H and AB=0 then A+B is also a tetrajection such that the null space of A+B is the intersection of the null spaces of A and B and the range of A+B is the direct sum of ranges of A and B.

Proof:

We have $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$

Now, $AB = 0 \Rightarrow (AB)^* = O^* \Rightarrow B^*A^* = 0 \Rightarrow BA = 0$

Therefore $AB = BA = 0$

Hence $(A+B)^2 = A^2 + B^2$

$(A+B)^3 = (A+B)(A+B)^2$

$(A+B)(A^2+B^2) = A^3 + AB^2 + BA^2 + B^3$

$= A^3 + (AB)B + (BA)A + B^3 = A^3 + B^3$

$(A+B)^4 = (A+B)(A+B)^3$

$= (A+B)(A^3+B^3) = A^4 + AB^3 + BA^3 + B^4$

$= A^4 + (AB)B^2 + (BA)A^2 + B^4$

$= A^4 + B^4$

$= A + B \; and \; (A+B)^* = A^* + B^* = A + B$

Hence $A + B$ is a tetrajection

Now we have to prove that

$N_{A+B} = N_A \cap N_B \; and \; R_{A+B} = R_A \oplus R_B$

We have $N_{A+B} = \{z: (A+B)z = 0\}$

$= \{z: Az + Bz = 0\}$

$= \{z: Az = -Bz\}$

Let $z \in N_{A+B}$. Since $AB = 0$

Therefore $0 = (AB)z = A(Bz) = A(-Az) = -A^2z$

Hence $A^2z = 0$

$\Rightarrow A^3z = A(A^2z) = A(0) = 0$

$\Rightarrow z \in N_A \; (since \; N_{A^3} = N_A)$

Again, since $BA = 0$, therefore

$0 = (BA)z = B(Az) = B(-Bz) = -B^2z$

$\Rightarrow B^2z = 0$

$\Rightarrow B^3z = B(B^2z) = B(0) = 0$

$\Rightarrow z \in N_B$

Therefore $z \in N_A \; and \; z \in N_B$

$\Rightarrow z \in N_A \cap N_B$

Thus $z \in N_{A+B} \Rightarrow z \in N_A \cap N_B$

Therefore $N_{A+B} \subseteq N_A \cap N_B$ ---------------------- 3.15

Let $z \in N_A \cap N_B$, then $z \in N_A \; and \; z \in N_B$

$\Rightarrow Az = 0 \; and \; Bz = 0$

$\Rightarrow Az + Bz = 0 + 0 = 0$

$\Rightarrow (A+B)z = 0$

$\Rightarrow z \in N_{A+B}$

Thus $z \in N_A \cap N_B \Rightarrow z \in N_{A+B}$

Therefore $N_A \cap N_B \subseteq N_{A+B}$ --------------------- (3.16)

Hence from (3.15) and (3.16)

We get, $N_{A+B} = N_A \cap N_B$

Let z be an element in R_{A+B}, then $(A+B)^3z = z \Rightarrow (A^3+B^3)z = z$

Now as A and B are tetrajections on H, so

$A^3z = A(A^2z) \in R_A \; and \; B^3z = B(B^2z) \in R_B$

Hence $z = A^3z + B^3z \in R_A + R_B$

Hence $R_{A+B} \subseteq R_A + R_B$ ------------------------ (3.17)

Conversely, let $z \in R_A + R_B, then \; we \; can \; write \; z = z_1 + z_2$

Such that $z_1 \in R_A \; and \; z_2 \in R_B$

Hence $A^3z_1 = z_1 and \; B^3z_2 = z_2$

Therefore $(A+B)^3z = (A^3+B^3)(z_1+z_2)$

$= A^3z_1 + A^3z_2 + B^3z_1 + B^3z_2$

$= A^3z_1 + A^3(B^3z_2) + B^3(A^3z_1) + B^3z_2$

$= z_1 + (AB)^3z_2 + (BA)^3z_1 + z_2$

$= z_1 + z_2 \;\; (as \; AB = 0, BA = 0)$

$= z$

Hence $z \in R_{A+B}$

Thus $z \in R_A + R_B \Rightarrow z \in R_{A+B}$

Therefore $R_A + R_B \subseteq R_{A+B}$ --------------------- (3.18)

From (3.17) and (3.18), we get

$R_{A+B} = R_A + R_B$

Now, let $z \in R_A \cap R_B$, then $z \in R_A \; and \; z \in R_B$

$z \in R_A \Rightarrow A^3z = z \; and \; z \in R_B \Rightarrow B^3z$

Therefore $z = A^3z = A^3(B^3z) = (A^3B^3)z = (AB)^3z = 0$

Hence $R_A \cap R_B = \{0\}$

Therefore $R_{A+B} = R_A \oplus R_B$

Remark

From theorem (3.I), we see that with the given conditions in the above theorem, AB is also a tetrajection.

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