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Study of a Polynomial I+aE+bE² in (𝝀,𝝁) Jection of Third Order

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ABSTRACT

In this paper, I study a polynomial of form I+aE+bE^2, where E is a (𝝀,𝝁)-jection; a,b being scalars. I investigate when I+aE+bE^2 is a (𝝀,𝝁)-jection.

Keywords:- (𝝀,𝝁)-jection, projection

Introduction

In his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces" [1], Dr. P. Chandra had defined a trijection operator by $E^3 = E$ where E is an operator. An operator E has been defined to be a projection if $E^2 = E$ as given in Dunford and Schwartz [2], p.37 or Rudin [3], p.126. I had defined previously E to be a 𝝀-jection (or 𝝀-jection of third order) [4], if
$E^3 + \lambda E^2 = (1 + \lambda)E, \lambda \text{ being a scalar}$
For further generalisation I define E to be a (𝝀,𝝁)-jection if
$\lambda E^3 + \mu E^2 = (\lambda + \mu)E, \lambda, \mu \text{ being scalars}$
Let us consider different values of 𝝀,𝝁 and corresponding operators.
When $\lambda = 1, \mu = 0$ then $E^3 = E$, a trijection
When $\lambda = 0, \mu = 1$ then $E^2 = E$, a projection
When $\lambda = 1, \mu = \lambda$ then E is a 𝝀-jection

Main Result

Theorem 1 Let E be a (𝝀,𝝁)-jection, then E can be expressed as a linear combination of two mutually orthogonal projections. (assuming $\boldsymbol{\lambda \neq 0, \lambda + \mu \neq 0, 2\lambda + \mu \neq 0}$)
Proof:- Let us assume that E can be expressed as a linear combination aL+bM of two mutually orthogonal projections L and M.
Thus $E = aL + bM$ where $L^2 = L, M^2 = M \text{ and } LM = 0$
So, $aE = a^2L + abM$ -- (1)

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Also, $E^2 = a^2L + b^2M$ --- (2)
Subtracting (2) from (1),
$aE - E^2 = (ab - b^2)M = b(a-b)M$
Hence $M = \frac{aE-E^2}{b(a-b)}$ assuming $(a \neq 0, b \neq 0, a \neq b)$
So, $bM = \frac{aE-E^2}{a-b}$
Hence $aL = E - bM = E - \frac{(aE-E^2)}{a-b} = \frac{E^2-bE}{a-b}$
$\Rightarrow L = \frac{E^2 - bE}{a(a-b)}$
But $LM = 0$
Hence $\frac{E^2-bE}{a(a-b)} * \frac{aE-E^2}{b(a-b)} = 0$
$\Rightarrow (E^2 - bE)(aE - E^2) = 0$
$\Rightarrow aE^3 - E^4 - abE^2 + bE^3 = 0$
$\Rightarrow E^4 = (a+b)E^3 - abE^2$ -- (3)
Since E is a (**λ**,**μ**)-jection,
$\lambda E^3 + \mu E^2 = (\lambda + \mu)E$
$\Rightarrow \lambda E^3 = (\lambda + \mu)E - \mu E^2$
$\Rightarrow \lambda E^4 = (\lambda + \mu)E^2 - \mu E^3$
$\Rightarrow E^4 = \frac{(\lambda+\mu)}{\lambda}E^2 - \frac{\mu}{\lambda}E^3$ (assuming $\lambda \neq 0$) -- (4)
Comparing (3) and (4),
$a + b = \frac{-\mu}{\lambda}$ *and* $ab = \frac{-(\lambda+\mu)}{\lambda}$
Hence $(a-b)^2 = (a+b)^2 - 4ab = \frac{\mu^2}{\lambda^2} + \frac{4(\lambda+\mu)}{\lambda}$
$= \frac{\mu^2 + 4\lambda^2 + 4\lambda\mu}{\lambda^2} = \frac{(\mu + 2\lambda)^2}{\lambda^2}$
So, $a - b = \frac{\mu+2\lambda}{\lambda}, but\ a + b = \frac{-\mu}{\lambda}$
Adding
$2a = \frac{2\lambda}{\lambda} = 2 \Rightarrow a = 1$
Then $b = \frac{-\mu}{\lambda} - a = \frac{-\mu}{\lambda} - 1 = \frac{-(\lambda+\mu)}{\lambda}$
Hence $E = L - \frac{(\lambda+\mu)}{\lambda}M$ (assuming $\lambda \neq 0$)
So, we have expressed E as a linear combination of two mutually orthogonal projections L and M.
Now we find values of L and M.
We have
$L = \frac{E^2-bE}{a(a-b)} = \frac{E^2+\frac{(\lambda+\mu)}{\lambda}E}{1+\frac{(\lambda+\mu)}{\lambda}} = \frac{\lambda E^2+(\lambda+\mu)E}{2\lambda+\mu}$ $(assuming\ 2\lambda + \mu \neq 0)$ ---------------------------- (5)
$M = \frac{aE-E^2}{b(a-b)} = \frac{E-E^2}{\frac{-(\lambda+\mu)}{\lambda}*(1+\frac{\lambda+\mu}{\lambda})} = \frac{E^2-E}{(\frac{\lambda+\mu}{\lambda})*(\frac{2\lambda+\mu}{\lambda})} = \frac{\lambda^2(E^2-E)}{(\lambda+\mu)(2\lambda+\mu)}$ $(assuming\ \lambda + \mu \neq 0)$ -------- (6)

Theorem 2

We investigate the cases when an expression of form I+aE+bE^2 is a (λ,μ)-jection, E being a (λ,μ)-jection, a,b being scalars. We find nine cases out of which four are projections.

Proof:-

As we have seen in theorem 1 that E can be put in the form

$E = L - \frac{(\mu+\lambda)}{\lambda}M \;\; where\ \lambda \neq 0$

Let $t = \frac{(\lambda+\mu)}{\lambda}\ then$

$E = L - tM$

Also $E^2 = L^2 + t^2M^2 - 2tLM = L + t^2M \;\; (Since\ L^2 = L, M^2 = M, LM = 0)$

So, $I + aE + bE^2 = I + a(L - tM) + b(L + t^2M)$

$= I + (a+b)L + (bt^2 - at)M$

$= I + xL + yM$

Where $x = a + b, y = bt^2 - at$

Hence

$(I + aE + bE^2)^2 = (I + xL + yM)^2$

$= I + x^2L + y^2M + 2xL + 2yM$

$= I + (x^2 + 2x)L + (y^2 + 2y)M$

$(I + aE + bE^2)^3 = (I + xL + yM)^3$

$= (I + xL + yM)^2(I + xL + yM)$

$= [I + (x^2 + 2x)L + (y^2 + 2y)M](I + xL + yM)$

$= [I + (x^2 + 2x)L + (y^2 + 2y)M + xL + x(x^2 + 2x)L] + yM + y(y^2 + 2y)M$

$= [I + (x^2 + 2x)L + (y^2 + 2y)M + xL + (x^3 + 2x^2)L] + yM + (y^3 + 2y^2)M$

$= I + (x^3 + 3x^2 + 3x)L + (y^3 + 3y^2 + 3y)M$

If I+aE+bE^2 is a (λ,μ)-jection, then

$\lambda(I + aE + bE^2)^3 + \mu(I + aE + bE^2)^2 = (\lambda+\mu)(I + aE + bE^2)$

$\Rightarrow \lambda[I + (x^3 + 3x^2 + 3x)L + (y^3 + 3y^2 + 3y)M] + \mu[I + (x^2 + 2x)L + (y^2 + 2y)M]$

$= (\lambda+\mu)(I + xL + yM)$

We see that co-efficients of I on both sides are same which is λ+μ.

Now compare co-efficients of L on both sides. Then

$\lambda(x^3 + 3x^2 + 3x) + \mu(x^2 + 2x) = (\lambda+\mu)x$ ---------------------------------- (7)

One obvious solution is $x = 0\ or\ a + b = 0$

Otherwise

$\lambda(x^2 + 3x + 3) + \mu(x + 2) = \lambda + \mu$

$\Rightarrow \lambda x^2 + x(3\lambda + \mu) + 2\lambda + \mu = 0$

$\Rightarrow (x + 1)(\lambda x + 2\lambda + \mu) = 0$

$\Rightarrow x = -1\ or\ x = -2 - \frac{\mu}{\lambda} = \frac{-(2\lambda+\mu)}{\lambda} = \frac{-(\lambda+\mu)}{\lambda} - 1 = -t - 1$

Comparing co-efficient of M on both sides,

$\lambda(y^3 + 3y^2 + 3y) + \mu(y^2 + 2y) = (\lambda+\mu)y$

Hence as before

$y = 0, -1, -t - 1$

So considering these values of x and y, we get 9 solutions. Now we discuss these solutions.

i. Consider the case $x = 0, y = 0$. Then

$a + b = 0$ and $bt^2 - at = 0$
Since $\lambda \neq 0, \lambda + \mu \neq 0, t = \frac{\lambda+\mu}{\lambda} \neq 0$
So, $bt^2 - at = 0 \Rightarrow bt - a = 0$
$\Rightarrow t = \frac{a}{b} = -1 \; since \; a + b = 0$
Thus $\frac{\lambda+\mu}{\lambda} = -1 \Rightarrow 2\lambda + \mu = 0, not\ permissible$
So take $a = 0, b = 0. Then\ I + aE + bE^2 = I$, the identity operator which is clearly a projection.

ii. Let $x = 0, y = -1$
i.e. $a + b = 0 \; and \; t(bt - a) = -1$
Since $b = -a, t(bt + b) = -1 \Rightarrow t(t + 1)b = -1$
$\Rightarrow \frac{\lambda + \mu}{\lambda} * \frac{(2\lambda + \mu)}{\lambda} b = -1$
$\Rightarrow b = \frac{-\lambda^2}{(\lambda + \mu)(2\lambda + \mu)}$
Hence $a = -b = \frac{\lambda^2}{(\lambda+\mu)(2\lambda+\mu)}$
So, $I + aE + bE^2 = I + \frac{\lambda^2(E-E^2)}{(\lambda+\mu)(2\lambda+\mu)}$
$= I - \frac{\lambda^2(E^2-E)}{(\lambda+\mu)(2\lambda+\mu)} = I - M$ (using equation 6)
Which is clearly a projection, since M is a projection.

iii. Let $x = 0, y = -t - 1$
Then $I + aE + bE^2 = I + xL + yM = I - (t + 1)M$
$= I - \frac{(2\lambda+\mu)}{\lambda} M = I - \frac{\lambda(E^2-E)}{\lambda+\mu}$ (using equation 6)

iv. Let $x = -1, y = 0, then$
$I + xL + yM = I - L$
Which is a projection, since L is a projection.

v. Let $x = -1, y = -1 \; then$
$I + xL + yM = I - (L + M) = I - N$
Where $N = L + M$. Since L,M are projections such that $LM = 0$, so N is also a projection.
Hence $I - N$ is also a projection.

vi. Let $x = -1, y = -t - 1 = \frac{-(2\lambda+\mu)}{\lambda}$
Then $I + xL + yM = I - L - \frac{(2\lambda+\mu)}{\lambda} M$

vii. Let $x = -(t + 1), y = 0$
Then $I + xL + yM = I - \frac{(2\lambda+\mu)}{\lambda} L$

viii. Let $x = -(t + 1), y = -1$
Then $I + xL + yM = I - \frac{(2\lambda+\mu)}{\lambda} L - M$

ix. Let $x = -(t + 1), y = -(t + 1)$
Then $I + xL + yM = I - (t + 1)(L + M) = I - (t + 1)N$
Thus we have discussed all nine cases in which i, ii, iv, v give projections. Thus four cases give projections.

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