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On a Special Type of Operator, Called $\boldsymbol{\lambda}$-jection of Fourth Order

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ABSTRACT
In this paper I have considered an operator called $\boldsymbol{\lambda}$-jection of fourth order. Forms of an operator E in R^2 which satisfy its conditions has been discussed.

Keywords:- $\boldsymbol{\lambda}$-jection of third order, $\boldsymbol{\lambda}$-jection of fourth order, projection, trijection

I. INTRODUCTION

A trijection operator E is defined as $E^3 = E$ by Dr.P.Chandra in his Ph.D. thesis titled "Investigation into the theory of operators and linear spaces".[1] A projection operator E has been defined as $E^2 = E$ in Dunford and Schwartz [2], p.37 and Rudin [3], p.126. In a previous paper of mine I have defined E to be a $\boldsymbol{\lambda}$-jection of third order [4] if

$E^3 + \boldsymbol{\lambda E^2} = (\mathbf{1} + \boldsymbol{\lambda})\boldsymbol{E}$ $\boldsymbol{\lambda}$ being a scalar.

In case $\boldsymbol{\lambda} = \mathbf{0}$, we have $E^3 = E$, a trijection.

To extend this idea further, we define E to be a $\boldsymbol{\lambda}$-jection of fourth order if

$E^4 + \boldsymbol{\lambda E^3} = (\mathbf{1} + \boldsymbol{\lambda})\boldsymbol{E^2}$,

$\boldsymbol{\lambda}$ being a projection. If E is a $\boldsymbol{\lambda}$-jection of third order, it is clearly a $\boldsymbol{\lambda}$-jection of fourth order but not conversely.

II. Main Result

We investigate when an operator E on R^2 happens to be a $\boldsymbol{\lambda}$-jection of fourth order.

Theorem 1

Let E be an operator defined on R^2 by $\boldsymbol{E(x,y) = (ax + by, cx + dy)}$

with a,b,c,d real numbers.

We find out conditions when E is a $\boldsymbol{\lambda}$-jection of fourth order.

Proof:-

We have $E^2(x,y) = E(E(x,y)) = E(ax + by, cx + dy)$

$= (a(ax + by) + b(cx + dy), c(ax + by) + d(cx + dy))$

$= ((a^2 + bc)x + b(a + d)y, c(a + d)x + (bc + d^2)y)$

$= (Ax + By, Cx + Dy), say$

where $A = a^2 + bc, B = b(a + d), C = c(a + d), D = bc + d^2$

Let $ad - bc = m \; and \; a + d = n.$

Then $d = n - a$

Hence $a(n - a) - bc = m$

$\Rightarrow an - a^2 - bc = m$

$\Rightarrow a^2 + bc = an - m, bc = an - m - a^2$

$So, A = an - m, B = bn, C = cn$

$D = bc + d^2 = an - m - a^2 + (n - a)^2$

$= an - m - a^2 + n^2 + a^2 - 2an = n^2 - an - m$

Now

$E^3(x,y) = E(E^2(x,y)) = E(Ax + By, Cx + Dy)$

$= (a(Ax + By) + b(Cx + Dy), c(Ax + By) + d(Cx + Dy))$

$= ((aA + bC)x + (aB + bD)y, (cA + dC)x + (cB + dD)y)$

$= (A_1x + B_1y, C_1x + D_1y), say$

Where

$A_1 = aA + bC = a(an - m) + bcn$

$= a^2n - am + n(an - m - a^2) = an^2 - mn - am$

$B_1 = aB + bD = abn + b(n^2 - an - m) = b(n^2 - m),$

$C_1 = cA + dC = c(an - m) + (n - a)cn = c(n^2 - m),$

and $D_1 = cB + dD = cbn + (n - a)(n^2 - an - m)$

$= (an - m - a^2)n + n^3 - an^2 - mn - an^2 + a^2n + am$

$= n^3 - an^2 - 2mn + am$

Also,

$E^4(x,y) = E^2(E^2(x,y)) = E^2(Ax + By, Cx + Dy)$

$= (A_2x + B_2y, C_2x + D_2y), say$

where $A_2 = A^2 + BC = (an - m)^2 + bc(a + d)^n$

$= a^2n^2 + m^2 - 2amn + (an - m - a^2)n^2$

$= an^3 - mn^2 - 2amn + m^2$

$B_2 = B(A + D) = bn(an - m + n^2 - an - m) = bn(n^2 - 2m)$

$C_2 = C(A + D) = cn(n^2 - 2m)$

$D_2 = BC + D^2 = bcn^2 + (n^2 - an - m)^2$

$= (an - m - a^2)n^2 + n^4 + a^2n^2 + m^2 - 2an^3 - 2mn^2 + 2amn$

$= n^4 - an^3 - 3mn^2 + 2amn + m^2$

Let $\lambda = \mu - 1$, then E is a $\boldsymbol{\lambda}$-jection of fourth order,

if $E^4 + (\mu - 1)E^3 = \mu E^2$

Let $(x, y) \in R^2, then$

$E^4(x, y) + (\mu - 1)E^3(x, y) = \mu E^2(x, y)$

$\Rightarrow (A_2x + B_2y, C_2x + D_2y) + (\mu - 1)(A_1x + B_1y, C_1x + D_1y)$

$-\mu(Ax + By, Cx + Dy) = 0$

Equating co-efficients of x,y from both co-ordinates,

$A_2 + (\mu - 1)A_1 - \mu A = 0$ -- (1)

$B_2 + (\mu - 1)B_1 - \mu B = 0$ -- (2)

$C_2 + (\mu - 1)C_1 - \mu C = 0$ -- (3)

$D_2 + (\mu - 1)D_1 - \mu D = 0$ -- (4)

Due to (1),

$(an^3 - mn^2 - 2amn + m^2) + (\mu - 1)(an^2 - mn - am) - \mu(an - m) = 0$

$\Rightarrow an^3 + n^2(a(\mu - 1) - m) - mn(2a + \mu - 1) - a\mu n$

$+m^2 - (\mu - 1)am + \mu m = 0$ -- (5)

From (2),

$bn(n^2 - 2m) + (\mu - 1)b(n^2 - m) - \mu bn = 0$

Let $b \neq 0$, then

$n(n^2 - 2m) + (\mu - 1)(n^2 - m) - \mu n = 0$ ------------------------------ (6)

$\Rightarrow n^3 - 2mn + (\mu - 1)n^2 - m(\mu - 1) - \mu n = 0$ --------------------- (7)

Now (5) can be put in form

$a(n^3 + (\mu - 1)n^2 - 2mn - \mu n - (\mu - 1)m) - mn^2 - (\mu - 1)mn$

$+m^2 + \mu m = 0$

Using (7),

$-mn^2 - (\mu - 1)mn + m^2 + \mu m = 0$

$\Rightarrow m[-n^2 - (\mu - 1)n + m + \mu] = 0$ ------------------------------------- (8)

From (3),

$cn(n^2 - 2m) + (\mu - 1)c(n^2 - m) - \mu cn = 0$

$Let\ c \neq 0, then$

$n(n^2 - 2m) + (\mu - 1)(n^2 - m) - \mu n = 0$

Which is same as (6)

From (4)

$(n^4 - an^3 - 3mn^2 + 2amn + m^2) + (\mu - 1)(n^3 - an^2 - 2mn + am)$

$-\mu(n^2 - an - m) = 0$

$\Rightarrow n^4 - 3mn^2 + m^2 + (\mu - 1)n^3 - 2mn(\mu - 1) - \mu n^2 + \mu m$

$-a[n^3 - 2mn + (\mu - 1)n^2 - m(\mu - 1) - \mu n] = 0$

Hence using (7),

$n^4 - 3mn^2 + m^2 + (\mu - 1)n^3 - 2mn(\mu - 1) - \mu n^2 + \mu m = 0$ ---------- (9)

From (8), taking m=0,

$n^4 + (\mu - 1)n^3 - \mu n^2 = 0$

$\Rightarrow n^2[n^2 + (\mu - 1)n - \mu] = 0$

$\Rightarrow n^2(n - 1)(n + \mu) = 0$

Hence $n = 0,1\ and\ -\mu$

Since we have considered $m = 0$, possibilities for (m,n) are (0,0), (0,1), (0,-μ)

From (8), let $m = n^2 + n(\mu - 1) - \mu$

Squaring

$m^2 = n^4 + n^2(\mu - 1)^2 + \mu^2 + 2n^3(\mu - 1) - 2n^2\mu - 2\mu(\mu - 1)n$

Substracting (9)

$m^2 = 3mn^2 - m^2 + (\mu - 1)n^3 + 2mn(\mu - 1) - n^2\mu - \mu m + n^2(\mu - 1)^2 + \mu^2$

$-2\mu(\mu - 1)n$

$= 3mn^2 - m^2 + (\mu - 1)n[n^2 + n(\mu - 1) - \mu] - \mu(\mu - 1)n + \mu^2 - \mu m$

$-n^2\mu + 2mn(\mu - 1)$

$= 3mn^2 - m^2 + mn(\mu - 1) - \mu(\mu - 1)n + \mu^2 - \mu m - n^2\mu + 2mn(\mu - 1)$

$= 3mn^2 - m^2 + 3mn(\mu - 1) - \mu(\mu - 1)n + \mu^2 - \mu m - n^2\mu$

$= 3m[n^2 + n(\mu - 1) - \mu] + 2m\mu - m^2 - \mu(\mu - 1)n + \mu^2 - n^2\mu$

$= 3m^2 + 2m\mu - m^2 - \mu[n^2 + n(\mu - 1) - \mu]$

$= 2m^2 + 2m\mu - \mu m$

$= 2m^2 + m\mu$

Thus $m^2 = 2m^2 + m\mu \Rightarrow m^2 + m\mu = 0$

$\Rightarrow m(m+\mu)=0$

$\Rightarrow m=0\ or\ -\mu$

We have already considered $m=0$

$m=-\mu$

$\Rightarrow n^2+n(\mu-1)-\mu=-\mu$

$\Rightarrow n^2+n(\mu-1)=0$

$\Rightarrow n\ [\ n+(\mu-1)\ ]=0$

$\Rightarrow n=0\ or\ 1-\mu$

Thus we have two possibilities $(-\mu,0)\ and\ (-\mu,1-\mu)$. Hence we get five possibilities.

$(0,0),(0,1),(0,-\mu),(-\mu,0)\ and\ (-\mu,1-\mu)$

Theorem 2

Let m=0, n=0, then

$\boldsymbol{E(x,y)=(ax+by,cx-ay)\ where\ bc=-a^2}$.

Also $\boldsymbol{E^2=0}$

Proof:-

When m=0, n=0,

$A=0,B=0,C=0,D=n^2-an-m=0$

$Hence\ E^2(x,y)=(Ax+By,Cx+Dy)=(0,0)$

So $E^2=0, Zero\ Operator$

$n=0\Rightarrow d=-a$

$bc=an-m-a^2=-a^2$

Hence $E(x,y)=(ax+by,cx-ay)\ where\ bc=-a^2$

Theorem 3

Let m=0, n=1, then

$\boldsymbol{E(x,y)=(ax+by,cx+(1-a)y)\ with\ bc=a-a^2}$

And E is a projection

Proof:-

$n=1\Rightarrow a+d=1\Rightarrow d=1-a$

$bc=an-m-a^2=a-a^2$

Hence $E(x,y)=(ax+by,cx+(1-a)y)\ where\ bc=a-a^2$

Also $A=a,B=b,C=c,D=n^2-an-m=1-a=d$

$So\ E^2(x,y)=(ax+by,cx+dy)=E(x,y)$

Hence E is a projection.

Theorem 4

Let m=0, n= -μ, then

$\boldsymbol{E(x,y) = (ax+by, cx-(\mu+a)y)}$ $\boldsymbol{where\ bc = -a\mu - a^2}$

Also $\boldsymbol{E^2 = -\mu E}$

Proof:-

Here $n = a + d = -\mu \Rightarrow d = -\mu - a = -(\mu + a)$

$bc = an - m - a^2 = -a\mu - a^2$

So, $E(x,y) = (ax + by, cx - (\mu + a)y)$ $where\ bc = -a\mu - a^2$

Hence $A = an - m = -a\mu, B = -b\mu, C = -c\mu$

$D = n^2 - an - m = \mu^2 + a\mu = \mu(a + \mu) = -d\mu$

Hence

$E^2(x,y) = (-a\mu x - b\mu y, -c\mu x - d\mu y)$

$= -\mu(ax + by, cx + dy) = -\mu E(x,y)$

Hence $E^2 = -\mu E$

Theorem 5

Let $\boldsymbol{m = -\mu, n = 0\ then}$

$\boldsymbol{E(x,y) = (ax+by, cx-ay)}$ **where $\boldsymbol{bc = \mu - a^2}$**

Also $\boldsymbol{E^2 = I}$

Proof:-

$n = 0 \Rightarrow a + d = 0 \Rightarrow d = -a$

$bc = an - m - a^2 = -(-\mu) - a^2 = \mu - a^2$

Hence

$E(x,y) = (ax + by, cx - ay)\ where\ bc = \mu - a^2$

Also $A = an - m = \mu, B = bn = 0, C = 0, D = n^2 - an - m = \mu$

Hence $E^2(x,y) = (\mu x + 0.y, 0.x + \mu y) = (\mu x, \mu y) = \mu I(x,y)$

Therefore

$E^2 = \mu I$

$\Rightarrow E^3 = \mu E \Rightarrow E^4 = \mu E^2 = \mu^2 I$

Substitute the values in

$E^4 + (\mu - 1)E^3 = \mu E^2$

We get

$\mu^2 I + (\mu - 1)\mu E = \mu.\mu I = \mu^2 I$

$\Rightarrow (\mu - 1)\mu E = 0$

Since in general $E \neq 0$,

$\mu(\mu - 1) = 0 \Rightarrow \mu = 0 \; or \; 1$

When $\mu = 0$, we have m=0, n=0 which we have discussed in theorem 2

Hence $\mu = 1, i.e. E^2 = I$

Theorem 6

Let $\boldsymbol{m = -\mu \; and \; n = 1 - \mu}$

Then $\boldsymbol{E(x, y) = (ax + by, cx + (1 - \mu - a)y)}$

where $\boldsymbol{bc = a - a\mu - a^2 + \mu}$

Also in this case,

$\boldsymbol{E^2 = (1 - \mu)E + \mu I}$

Proof:-

$n = a + d = 1 - \mu \Rightarrow d = 1 - \mu - a$

and $bc = an - m - a^2 = a(1 - \mu) + \mu - a^2 = a - a\mu - a^2 + \mu$

$A = an - m, B = bn, C = cn$

$D = n^2 - an - m = n(n - a) - m = nd - m$

So,

$E^2(x, y) = ((an - m)x + bny, cnx + (nd - m)y)$

$= (anx + bny, cnx + dny) - m(x, y)$

$= n(ax + by, cx + dy) - m(x, y)$

$= nE(x, y) - m(x, y)$

Putting $n = 1 - \mu \; and \; m = -\mu$

$E^2(x, y) = (1 - \mu)E(x, y) + \mu I(x, y)$

Hence $E^2 = (1 - \mu)E + \mu I$

Theorem 7

Let $\boldsymbol{b = 0, c = 0, then \; E(x, y) \; has \; nine \; possibilities,}$

$$\boldsymbol{(0, 0), (0, y), (0, -\mu y), (x, 0), (x, y), (x, -\mu y), (-\mu x, 0), (-\mu x, y) \; and \; (-\mu x, -\mu y)}$$

Proof:-

Let $b = c = 0$, then

$E(x, y) = (ax, dy)$

So, $E^2(x, y) = (a^2x, d^2y), E^3(x, y) = (a^3x, d^3y)\ and\ E^4(x, y) = (a^4x, d^4y)$

Hence $E^4(x, y) + (\mu - 1)E^3(x, y) = \mu E^2(x, y)$

$\Rightarrow (a^4x, d^4y) + (\mu - 1)(a^3x, d^3y) = \mu(a^2x, d^2y)$

$\Rightarrow (a^4x + (\mu - 1)a^3x - \mu a^2x, d^4y + (\mu - 1)d^3y - \mu d^2y) = 0$

Considering first co-ordinate, taking co-efficient of x to be 0,

$a^4 + (\mu - 1)a^3 - \mu a^2 = 0$

$\Rightarrow a^2[a^2 + (\mu - 1)a - \mu] = 0$

$\Rightarrow a^2(a - 1)(a + \mu) = 0$

$\Rightarrow a = 0, 1, -\mu$

Since expression for d is similar

$d = 0, 1, -\mu$

Considering all values of a and d, we have nine possibilities which are given by

$E(x, y) = (0,0), (0, y), (0, -\mu y), (x, 0), (x, y), (x, -\mu y), (-\mu x, 0),$

$(-\mu x, y)\ and\ (-\mu x, -\mu y)$

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