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Residue Inverse Laplace Transforms in Complex

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ABSTRACT:
The application of Laplace transformation in determination Initial value Problems (IVP's) of Ordinary Differential Equations (ODE's) of order n, $n \epsilon \mathbb{Z}+$ is renowned to students. The inversion of Laplace transformation in determination initial value problems of ODE's by the standard pure mathematics methodology (i.e. through resolving into partial fraction and the use of Laplace Transforms table) can be very tedious and time overwhelming, particularly once the Laplace transforms table is not readily available, thus renders the man of science unfit. In this paper, , we've got reviewed the standard pure mathematics methodology and currently show however the Residue Theorem of complex analysis will best be applied on to get the inverse Laplace transform which circumvents the rigor of resolving into partial fraction and also the use of Laplace transforms table which normally resolve into resultant time wastage as continuously the case with the traditional method.
Keywords : Laplace transforms, Fourier Transforms, Residue theorem, Jordan's Lemma

INTRODUCTION

It is well-known that function $\$f(s) = s^\wedge \left\{-\frac{1}{2}\right\}\$$ is invariant to Laplace and Fourier transforms(to within a scale factor).Proving that this is the case, at least for Laplace transforms, is far from trival. I define the computation below; the previous computation can function a guide.
This integral may be attacked with the residue theorem, but not the usual way for these inverse Laplace transforms. Basically ,the Bromwich contour must not include the branch point at the origin.The result is then a hole contour that goes up and back round the negative real axis and encircles the origin from $\arg{s}=\pi$ to $\arg{$}=-\pi$.
Consider

$$\oint_c ds \frac{e^{st}}{\sqrt{s}}$$

Where C is the above-described contour . By the residue theorem (or Cauchy's integral theorem),this integral is zero because there are no poles within C.C, however , has 6 pieces (see the previous post):the original integral along $\Re{s}=a$, a first circular arc of large radius R above the negative real axis, a section that goes in a positive direction just above the negative real axis a circular arc of small radius r around the origin , a second circular arc of large radius R below the negative real axis, a section that goes in a positive direction simply on top of the negative real axis, and another section just under the negative real axis during a negative direction.

Explanation

The Laplace transform:
for a function (signal) *f(t)* which is zero for *t<0*, the Laplace transform is

$$F(s) = \int_0^\infty f(t)\, e^{-st} dt$$

Here we use $s = \sigma + j\omega$ in place of $z = x + jy$, so we have

$$F(s) = \int_0^\infty (f(t)\, e^{-\sigma t}) e^{-j\omega t} dt$$

Problem: Given *F(s)* how do we obtain *f(t)*?

The Fourier transform:

The Fourier transform of *f(t)* is

$$\hat{f}(w) = \int_{-\infty}^\infty f(t)\, e^{-j\omega t} dt$$

and the inverse transform is

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^\infty \hat{f}(w)\, e^{-j\omega t} dt$$

In the Laplace transforms Let ,

$$\emptyset(t) = f(t) e^{-\sigma t} \ , \qquad t \geq 0$$
$$= 0 \ \ , t < 0$$

Where σ is a constant .Taking the Fourier transform of $\emptyset(t)$:

$$\widehat{\emptyset}(w) = \int_0^\infty \emptyset(t)\, e^{-j\omega t} dt$$
$$= \int_0^\infty f(t)\, e^{-\sigma t} e^{-j\omega t} dt$$
$$= F(\sigma + j\omega)$$

So, taking the inverse Fourier transform:

$$\emptyset(t) = f(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{\omega=-\infty}^\infty F(\sigma + j\omega)\, e^{j\omega t} d\omega$$

Hence,

$$f(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^\infty F(\sigma + j\omega)\, e^{j\omega t} e^{\sigma t} d\omega$$

Let $s = \sigma + j\omega$, $s = jd\omega$, then

$$f(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st}\, ds \, , t \geq 0$$
$$= 0 \quad , \quad t < 0$$

The inverse Laplace transform:

The formula for the inverse Laplace transform was obtained in the previous section as:

$$f(t) = \frac{1}{2\pi j} \int_{s=\sigma-j\infty}^{\sigma+j\infty} F(s) e^{st}\, ds$$

The relevant questions here are :

1.How do we choose the real parameter σ ?

2.How do we evaluate the integral ?

We already know that *f(t)=0* ,*t<0,* and we shall see that this gives us an answer to (1).

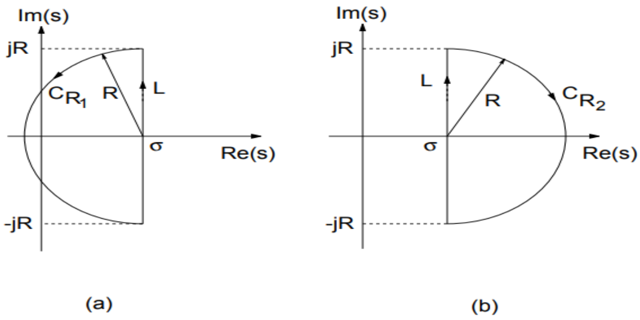

Figure 1: Integration contours

First , consider the closed contour $C_1 = C_{R_1} + L$ shown in Figure 1a.Then ,

$$\oint_{C_1} F(S)\, e^{st}\, ds = \int_L F(S)\, e^{st}\, ds + \int_{C_{R_1}} F(S)\, e^{st}\, ds$$

$$= \int_{\sigma - jR}^{\sigma + jR} F(s) e^{st}\, ds + \int_{C_{R_1}} F(S)\, e^{st}\, ds$$

$$= 2\pi j \sum_{poles\ in\ C_1} Res\,[F(S)e^{st}]$$

Note that the residues are at the poles inside C_1. Now , as $R \to \infty$, $\int_{\sigma-jR}^{\sigma+jR} F(s)e^{st}\, ds$ becomes the integral we require, and we can show(Jordan's lemma,see appendix) that for $t > 0, \int_{C_{R_1}} F(s)\, e^{st} ds \to 0\ as\ \ R \to \infty$ $.So\ , for\ t > 0,$

$$\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds = \sum_{poles\ left\ of \sigma} Res\,[F(S)e^{st}]$$

Next consider the contour $C_2 = C_{R_2} + L$ shown in Figure 1b.As before,

$$\oint_{C_2} F(s) e^{st}\, ds = \int_L F(s)\, e^{st} ds + \int_{C_{R_2}} F(S)\, e^{st}\, ds$$

$$= \int_{\sigma - jR}^{\sigma + jR} F(s) e^{st}\, ds + \int_{C_{R_2}} F(S)\, e^{st}\, ds$$

$$= -2\pi j \sum_{poles\ in\ C_2} Res\,[F(S)e^{st}]$$

Note that the residues are at the poles inside C_2 and the minus sign is due to the fact that we are dealing with a clockwise contour.Again , using Jordan's lemma, we have that for $t < 0$,
$\int_{C_{R_1}} F(s)\, e^{st} ds \to 0\ as\ \ R \to \infty\ \ .So\ , for\ t < 0,$

$$\frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds = -\sum_{poles\ right\ of \sigma} Res\,[F(S)e^{st}]$$

But we know that this must be zero ,since f(t)=0,$t < 0$.Hence σ must be chosen such that C_2 does not contain any poles of $F(s)e^{st}$ $(as\ R \to \infty$),and thus C_1 must contain all poles of $F(s)e^{st}$.This is the answer to question (1).Note, finally, that since e^{st} is analytic everywhere (i.e. has no poles), the poles of $F(s)e^{st}$ are the same as the poles of $F(s)$. This answers question (2) and we have

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds = \sum_{all\ poles\ of\ F(s)} Res\,[F(S)e^{st}]$$

Example 1:

Find the inverse transform of $F(s) = \frac{1}{s+a}$.

The function $\frac{e^{st}}{s+a}$ has a simple pole at $s=-a$.Hence

$$Res_s=-a[F(s)e^{st}]=\lim_{s\to -a}[(s+a)\frac{e^{st}}{s+a}]=e^{-at}$$

and so,

$$L^{-1}\left[\frac{1}{s+a}\right]=e^{-at},t\geq 0$$
$$=0\ ,\quad t<0$$

Example 2 :

Find the inverse transform of $F(s)=\frac{1}{(s+a)^2}$.

In this case the function $\frac{e^{st}}{(s+a)^2}$ has a pole of order 2 at $s=-a$. Remember that if a function f(z) has a pole of order n, then its residue at this pole is given by

$$c_1=\frac{1}{(n-1)!}\lim_{z\to a}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n f(z)]$$

In our case the required residue is:

$$\frac{1}{(2-1)!}=\lim_{s\to -a}\frac{d}{ds}[(s+a)^2\frac{e^{st}}{(s+a)^2}]=\lim_{s\to -a}[te^{st}]=te^{-at}$$

and so,

$$L^{-1}\left[\frac{1}{(s+a)^2}\right]=te^{-at},t\geq 0$$
$$=0\ ,\quad t<0$$

Example 3:

Find the inverse transform of $F(s)=\frac{1}{(s+a)^2(s+b)}$.

In this case the function $\frac{e^{st}}{(s+a)^2(s+b)}$ has a pole of order 2 at $s=-a$ and a simple pole at $s=-b$. The residue at $s=-b\ is$

$$\lim s\to -b\left[\frac{e^{st}}{(s+a)^2}\right]=\frac{e^{-bt}}{(a-b)^2}$$

The residue at $s=-a$ is

$$\frac{1}{(2-1)!}=\lim\frac{d}{ds}\left[\frac{e^{st}}{(s+b)}\right]=\lim_{s\to -a}\left[\frac{te^{st}}{(s+b)}-\frac{e^{st}}{(s+b)^2}\right]=\frac{te^{-at}}{b-a}-\frac{e^{-at}}{(b-a)^2}$$

and so

$$L^{-1}\left[\frac{1}{(s+a)^2(s+b)}\right]=\frac{e^{-bt}}{(a-b)^2}+\frac{te^{-at}}{b-a}-\frac{e^{-at}}{(b-a)^2}\quad ,\ t\geq 0$$

Appendix : Jordan's Lemma

In the theory for inverting Laplace transforms using residue theory , we need the following result:

$$\lim_{R\to\infty}\int_{C_R}F(s)e^{st}\,ds=0$$

Where $t>0$ and C_R is the semicircular contour C_{R_1}shown in Figure 1a.Points on this contour are given by

$$s=\sigma+Re^{j\theta},\frac{\pi}{2}\leq\theta\leq\frac{3\pi}{2}$$

Looking at the table of standard transforms it can be seen that most statisfy the conditions $F(s)\to 0\ as\ |s|\to\infty(e.g.\frac{1}{s},\frac{1}{s+a},etc)$.Therefore on $C_R\ as\ R\to\infty, F(s)\to 0$.This means that for any $\epsilon>0$ as R can be found such that $|F(s)|=\left|F(\sigma+Re^{j\theta})\right|<\epsilon$. For this R we have

$$\left|\int_{C_R}F(s)e^{st}\,ds\right|\leq\epsilon\int_{C_R}|e^{st}|\,ds$$

On C_R,

$$|e^{st}| = \left|e^{\sigma+Re^{j\theta}}\right| = \left|e^{(\sigma+R\cos\theta+jR\sin\theta)t}\right| = \left|e^{(\sigma+R\cos\theta)t}\right|$$

And $ds = jRe^{j\theta}d\theta$.Therefore,

$$\epsilon\int_{C_R}|e^{st}|\,ds = \epsilon Re^{\sigma t}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{Rt\cos\theta}\,d\theta = 2\epsilon Re^{\sigma t}\int_0^{\frac{\pi}{2}} e^{-Rt\sin\theta}\,d\theta$$

Plotting graphs of $y - \sin\theta$ and the straight line $y = \frac{2}{\pi}\theta$, we see that

$$\sin\theta \geq \frac{2}{\pi}\theta\ , \qquad 0 \leq \theta \leq \frac{\pi}{2}$$

Hence the integral is less than

$$2R \in e^{\sigma t}\int_0^{\frac{\pi}{2}} e^{-\frac{2Rt}{\pi}\theta}\,d\theta = \frac{\in \pi e^{\sigma t}}{t}(1 - e^{-Rt})$$

For any $t > 0$ this last quantity goes to zero as $R \to \infty$ (because $\in$ also goes to zero),and we have therefore proved (1).

Conclusion

In comparing the methods of finding the inverse Laplace transform from the Residue inversion approach and the traditional method of resolving into partial fraction with the use of tables, both results are exact and valid. However, the method by Residue inversion is more direct, precise, efficient, time saving and has no need of resolving into partial fraction nor referring to Laplace transform tables at anytime for the complete solution of the ODE. Also, the residue inversion approached has to its advantage a fast and direct way of obtaining the solution of the ODE without the rigor of solving for the Homogeneous solution first, then finding the particular solution before applying the initial conditions in other to get the final solution of the ODE as it is the case with other algebraic methods of solving linear ODE's.

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