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Spectral Resolution of An Operator E², E Being A (5,2)-Jection

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ABSTRACT

In this paper we consider an operator E, called (5,2 -Jection on a Hilbert space. We obtain spectral resolution of E^2

Keywords :- (5,2)-Jection, Projection. AMS COassification number (2023)

I. INTRODUCTION

Dr. P. Chandra defined a trijection operator in his Ph.D. Thesis titled "INVESTIGATION INTO THE THEORY OF OPERATORS AND LINEAR SPACES" [1]. In Dunford and Schwarz [2], p. 37 and Rudin [3], p.126 a projection operator E has been defined as E^2=E. E is a trijection operator if E^3=E. Dr. Rajiv Kumar Mishra [4] has defined E to be a tetrajection if E^4=E. To generalise it, I have defined E to be a (5,2)-Jection if E^5=E^2. Thus every testrajection is a (5,2)-jection but not conversely.

II. DEFINITION

Let H be a Hilbert space and E an operator on H. Let $\pi_1, \pi_2, \ldots \ldots \ldots \ldots \ldots \ldots .. \pi_m$, be eigen values of E and $M_1, M_2, \ldots \ldots \ldots \ldots . M_m$, be their corresponding eigen spaces. Let $P_1, P_2, \ldots \ldots \ldots \ldots \ldots \ldots . P_m$, be the projections on these eigen spaces. Then according to definition of spectral theorem in Simmons [5], p.279-290, the following statements are all equivalent to one another.

I. The M'is are pairwise orthogonal and span H.

II. The P'is are pairwise orthogonal, $I = \sum_{i=1}^{m} P_i$ and $E = \sum_{i=1}^{m} \lambda_i P_i$

III. E is normal.

Then the set of eigen values of E is called its spectrum and is denoted by $\sigma(E)$. Also

$$E = \lambda_1 P_1 + \lambda_2 P_2 + \ldots\ldots\ldots\ldots + \lambda_m P_m$$

Expression for E given above is called the spectral resolution of E.

III. MAIN RESULT

Theorem I

Let E be a (5,2)-Jection on a Hilbert space H. Then E^2 can be expressed as a linear combination of three pairwise orthogonal projections.

Proof :-

Let E^2 can be expressed as

$$E^2 = aP_1 + bP_2 + cP_3 \ldots\ldots\ldots\ldots(1)$$

where a,b,c are scalars, P_1, P_2, P_3 are pairwise orthogonal projections, i.e.

$$P_1^2 = P_1, P_2^2 = P_2, P_3^2 = P_3$$

and, $P_1P_2 = P_2P_3 = P_3P_1 = 0$

Hence squaring both sides in (1),

$$E^4 = a^2P_1 + b^2P_2 + c^2P_3 \ldots\ldots\ldots\ldots(2)$$

Taking cube of both sides in (1),

$$E^6 = a^3P_1 + b^3P_2 + c^3P_3$$

Since $P_1^3 = P_1, P_1^2P_2 = P_1(P_1P_2) = P_1 0 = 0$ etc.

But $E^5 = E^2 \Rightarrow E^6 = E^3$, hence

$$E^3 = a^3P_1 + b^3P_2 + c^3P_3 \ldots\ldots\ldots\ldots(3)$$

From equations (1), (2), (3) we solve P_1, P_2, P_3 in term of E^2, E^3, E^4 and get

$P_1 = \frac{bcE^2 + E^3 - (b+c)E^4}{a(b-a)(c-a)}$ assuming a, b, c all different.

$P_2 = \frac{acE^2 + E^3 - (a+c)E^4}{b(a-b)(c-b)}$

and $P_3 = \frac{abE^2 + E^3 - (a+b)E^4}{c(a-c)(b-c)}$

Since $P_1P_2 = 0$

$$\left[\frac{bcE^2 + E^3 - (b+c)E^4}{a(b-a)(c-a)}\right] \left[\frac{acE^2 + E^3 - (a+c)E^4}{b(a-b)(c-b)}\right] = 0$$

$\Rightarrow [bcE^2 + E^3 - (b+c)E^4]\,[acE^2 + E^3 - (a+c)E^4] = 0$

We multiply the terms and noting that $E^6 = E^3, E^7 = E^4, E^8 = E^5 = E^2$,

We get

$E^2[c(a+b) + (a+c)(b+c)] + E^3[1 - abc - bc^2 - abc - ac^2]$

$+E^4[abc^2 - (a+c) - (b+c)] = 0$

Equating co-efficients of $E^2, E^3 and\ E^4 to\ 0$,

$$\left.\begin{aligned}&(a+c)(b+c)+c(a+b)=0\\&1-2abc-c^2(a+b)=0\\&abc^2-(a+c)-(b+c)=0\end{aligned}\right\}\quad(4)$$

$P_1P_3 = 0 \Rightarrow$

$[bcE^2 + E^3 - (b+c)E^4][abE^2 + E^3 - (a+b)E^4] = 0$

As before we multiply the terms and equate co-efficient of E^2, E^3 and E^4 to 0 and get

$$\left.\begin{aligned}&b(a+c)+(a+b)(b+c)=0\\&1-2abc-b^2(a+c)=0\\&ab^2c-(a+b)-(b+c)=0\end{aligned}\right\}\quad(5)$$

If we consider $P_2P_3 = 0$, as before multiplying and equating co-efficient of E^2, E^3, E^4 to 0, get

$$\left.\begin{aligned}&a(b+c)+(a+b)(a+c)=0\\&1-2abc-a^2(b+c)=0\\&a^2bc-(a+b)-(a+c)=0\end{aligned}\right\}\quad(6)$$

From equations (4), (5) and (6),

$(a+c)(b+c) + c(a+b) = 0$

$(a+b)(b+c) + b(a+c) = 0$

$(a+b)(a+c) + a(b+c) = 0$

Multiplying these equations by (a+b), (a+c) and (b+c) respectively and adding

$(a+c)(b+c)(a+b) + c(a+b)^2 = 0$

$(a+b)(b+c)(a+c) + b(a+c)^2 = 0$

$(a+b)(a+c)(b+c) + c(a+b)^2 = 0$

$Adding : 3(a+b)(b+c)(c+a) + 6abc + \sum a^2\,(b+c) = 0$

$\Rightarrow 3(a+b)(b+c)(c+a) + 4abc + 2abc + \sum a^2\,(b+c) = 0$

$\Rightarrow 3(a+b)(b+c)(c+a) + 4abc + (a+b)(b+c)(c+a) = 0$

$\Rightarrow 4(a+b)(b+c)(c+a) + 4abc = 0$

$\Rightarrow (a+b)(b+c)(c+a) = -abc$……………………………..(7)

Again from (4), (5) and (6),

$1 - 2abc - c^2(a+b) = 0$

$1 - 2abc - b^2(a+c) = 0$

$1 - 2abc - a^2(b+c) = 0$

Adding $3 - 6abc - \sum a^2(b+c) = 0$

$\Rightarrow 3 - 4abc - 2abc - \sum a^2(b+c) = 0$

$\Rightarrow 3 - 4abc - (a+b)(b+c)(c+a) = 0$

$\Rightarrow (a+b)(b+c)(c+a) = 3 - 4abc$..............................(8)

Considering equations (7) and (8),

$-abc = 3 - 4abc$

$\Rightarrow 3abc = 3 \Rightarrow abc = 1$

From equation (4)

$(a+c) + (b+c) = abc^2 = (abc)c = c$

$\Rightarrow a + b + c = 0$

Due to equation (5),

$(a+b) + (b+c) = ab^2c = b(abc) = b$

$\Rightarrow \sum a = 0$

In a same way due to equation (6),

$$\sum a = 0$$

Hence, $\sum a = 0$

Due to (4),

$1 - 2abc - c^2(a+b) = 0 \qquad (\sum a = 0)$

$\Rightarrow 1 - 2 - c^2(-c) = 1 - 2 + c^3 = c^3 - 1 = 0$

$\Rightarrow c^3 = 1$

Due to (5),

$1 - 2abc - b^2(a+c) = 0$

$\Rightarrow 1 - 2 - b^2(-b) = b^3 - 1 = 0$

$\Rightarrow b^3 = 1$

From a similar equation in (6), $a^3 = 1$

So a, b, c take values $1, \omega$ and ω^2. Moreover, since $\sum a = 0$,

let us choose $a = 1, b = \omega$ and $c = \omega^2$

Using these values, we find that

$$P_1 = \frac{E^2 + E^3 + E^4}{1(\omega - 1)(\omega^2 - 1)} = \frac{1}{3}(E^2 + E^3 + E^4)$$

$$P_2 = \frac{\omega^2 E^2 + E^3 + \omega E^4}{\omega(1-\omega)(\omega^2 - \omega)} = \frac{\omega^2 E^2 + E^3 + \omega E^4}{\omega^2(1-\omega)(\omega - 1)} = \frac{1}{3}(\omega^2 E^2 + E^3 + \omega E^4)$$

$$P_3 = \frac{\omega E^2 + E^3 + \omega^2 E^4}{\omega^2(1-\omega^2)(\omega - \omega^2)} = \frac{\omega E^2 + E^3 + \omega^2 E^4}{(1-\omega^2)(1-\omega)} = \frac{1}{3}(\omega E^2 + E^3 + \omega^2 E^4)$$

Hence

$E^2 = P_1 + \omega P_2 + \omega^2 P_3$

Where P_1, P_2, P_3 are pairwise orthogonal projections.

Also due to (2),

$E^4 = P_1 + \omega^2 P_2 + \omega P_3$

and due to (3),

$E^3 = P_1 + P_2 + P_3$

<u>Theorem II</u>

Let E be a (5,2)-Jection on a Hilbert space H. Then, there are four pairwise orthogonal projections P_1, P_2, P_3 and P_4 such that

$E^2 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4$

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are scalars and

$I = P_1 + P_2 + P_3 + P_4$

Proof :-

We have seen in <u>theorem I</u> that

$$E^2 = P_1 + \omega P_2 + \omega^2 P_3$$

Where, P_1, P_2 and P_3 are pairwise orthogonal projections. Let $Q = P_1 + P_2 + P_3$

Then $Q^2 = P_1^2 + P_2^2 + P_3^2 + 2P_1P_2 + 2P_2P_3 + 2P_3P_1 = P_1 + P_2 + P_3$

So Q is also a Projection

Also $P_1 + P_2 + P_3 = E^3$

Thus, $Q = E^3$. Hence I-Q is also a projection.

Let $P_4 = I - Q = I - P_1 - P_2 - P_3 = I - E^3$

ThenP_4 is a projection such that

$P_1P_4 = P_1(I - P_1 - P_2 - P_3) = P_1 - P_1^2 - P_1P_2 - P_1P_3 = P_1P_3 - P_1 - P_1 = 0$

Similarly $P_2P_4 = P_3P_4 = 0$

Hence P_1, P_2, P_3, P_4 are pairwise orthogonal.

Moreover,

$$P_1 + P_2 + P_3 + P_4 = Q + I - Q = I$$

Choose $\lambda_1 = 1, \lambda_2 = \omega, \lambda_3 = \omega^2, \lambda_4 = 0$, then

$E^2 = P_1 + \omega P_2 + \omega^2 P_3 + 0P_4 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4$ and $I = P_1 + P_2 + P_3 + P_4$

<u>Theorem III</u>

Range of Projection P_1 denoted by R_{P1} is given by

$R_{P1} = \{z: E^2 z = z\} = M_1$ (Say)

Proof :-

Let $z \epsilon R_{P1}$, then since P_1 is a projection, $P_1 z = z$

Now, $E^2 P_1 = E^2 \frac{1}{3}(E^2 + E^3 + E^4) = \frac{1}{3}(E^4 + E^2 + E^3) = P_1$

So, $E^2 z = E^2 P_1 z = P_1 z = z$ i.e. $z \epsilon M_1$

Hence, $R_{P1} \subseteq M_1$……………………….(9)

Conversely, Let $z\epsilon M_1$ i.e. $E^2z = z$

Then $E^4z = E^2(E^2z) = E^2z = z$

$$E^6z = E^2(E^4z) = E^2z = z \Rightarrow E^3z = z$$

Hence, $P_1z = \frac{1}{3}(E^2 + E^3 + E^4)z = \frac{1}{3}(z + z + z) = z$

So, $z\epsilon R_{P1}$

Hence $M_1 \subseteq R_{P1}$……………………….(10)

From (9) and (10),

$R_{P1} = M_1$

<u>Theorem IV</u>

$R_{P2} = \{z: E^2z = \omega z\} = M_2$ (Say).

Proof :-

Let $z\epsilon R_{P2}$ then $P_2z = z$.

Now $E^2P_2 = E^2\frac{1}{3}(\omega^2E^2 + E^3 + \omega E^4) = \frac{1}{3}(\omega^2E^4 + E^2 + \omega E^3)$

$= \frac{\omega}{3}(\omega^2E^2 + E^3 + \omega E^4) = \omega P_2$

Hence, $E^2z = E^2P_2z = \omega P_2z = \omega z$

So $z\epsilon M_2$

Thus, $R_{P2} \subseteq M_2$…………………………………(11)

Let $z\epsilon M_2$, then $E^2z = \omega z$

Hence, $E^4z = E^2(E^2z) = E^2(\omega z) = \omega E^2z = \omega^2z$

and $E^6z = E^2(E^4z) = E^2(\omega^2z) = \omega^2E^2z = \omega^2.\omega z = z$

But $E^6 = E^3$.Hence $E^3z = z$.

Then $P_2z = \frac{1}{3}(\omega^2E^2z + E^3z + \omega E^4z)$

$= \frac{1}{3}(\omega^2.\omega z + z + \omega.\omega^2z) = z.$

So $z\epsilon R_{P2}$

Thus $M_2 \subseteq R_{P2}$……………………..(12)

Due to (11) and (12),

$$R_{P2} = M_2$$

<u>Theorem V</u>

$R_{P3} = \{z: E^2z = \omega^2z\} = M_3$ (Say)

Proof :-

Let $z\epsilon R_{P3}$ then $P_3z = z$.

Now, $E^2P_3 = E^2\frac{(\omega E^2+E^3+\omega^2E^4)}{3} = \frac{1}{3}(\omega E^4 + E^2 + \omega^2E^3)$

$= \frac{1}{3}\omega^2(\omega E^2 + E^3 + \omega^2E^4) = \omega^2P_3$

Hence, $E^2z = (E^2(P_3)z) = (E^2P_3)z = \omega^2P_3z = \omega^2z$.

so, $z\epsilon M_3$

Hence $R_{P3} \subseteq M_3$……………………………………(13)

Conversely, let $z\varepsilon M_3$ i.e. $E^2z = \omega^2z$.

Then $E^4z = E^2(E^2z) = E^2(\omega^2z) = \omega^2E^2z = \omega^2.\omega^2z = \omega z$

$E^6z = E^2(E^4z) = E^2(\omega z) = \omega E^2z = \omega.\omega^2z = z$

$\Rightarrow E^3z = z$

Hence $P_3z = \frac{\omega E^2z+E^3z+\omega^2E^4z}{3} = \frac{\omega.\omega^2z+z+\omega^2.\omega z}{3} = z$

i.e. $z\varepsilon R_{P3}$

Hence $M_3 \subseteq R_{P3}$………………………………(14)

Due to (13) and (14),

$$R_{P3} = M_3$$

Theorem VI

$R_{P4} = \{z: E^2z = 0\} = M_4$ (Say)

Proof :-

We know that $P_4 = I - E^3$

Let $z\epsilon R_{P4}$ then $P_4z = z$

Then $E^2z = E^2P_4z = E^2(I - E^3)z = (E^2 - E^5)z = (E^2 - E^2)z = 0.$

So $z\epsilon M_4$

Hence $R_{P4} \subseteq M_4$……………………………..(15).

Let $z\epsilon M_4$ then $E^2z = 0 \Rightarrow E^3z = 0$

Hence $P_4z = (I - E^3)z = z - E^3z = z - 0 = z$

so $z\epsilon R_{P4}$

Hence $M_4 \subseteq R_{P4}$…………………………..(16)

From (15) and (16),

$$R_{P4} = M_4$$

Theorem VII

Let E be a (5,2)-Jection on a Hilbert space H. Let $\lambda_1 = 1$, $\lambda_2 = \omega$, $\lambda_3 = \omega^2$ and $\lambda_4 = 0$ be eigen values of E^2. Let M_1, M_2, M_3, M_4 be their corresponding eigen spaces. Let P_1, P_2, P_3 and P_4 be projections on these eigen spaces, where

$P_1 = \frac{1}{3}(E^2 + E^3 + E^4)$, $P_2 = \frac{1}{3}(\omega^2E^2 + E^3 + \omega E^4)$

$P_3 = \frac{1}{3}(\omega E^2 + E^3 + \omega^2E^4)$, $P_4 = I - E^3$

Then, $P_1 + P_2 + P_3 + P_4 = I$,

$P_i's$ are pairwise orthogonal and spectral resolution of E^2 is given by

$E^2 = \lambda_1P_1 + \lambda_2P_2 + \lambda_3P_3 + \lambda_4P_4$

Where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are given as above and spectrum of E^2 is given by

$\sigma(E^2) = \{0,1,\omega,\omega^2\}$

Proof :-

Theorems 2,3,4,5 and 6 show that $\lambda_1 = 1, \lambda_2 = \omega, \lambda_3 = \omega^2$ *and* $\lambda_4 = 0$ are eigen values of E^2, M_1, M_2, M_3, M_4 are their corresponding eigen spaces and P_1, P_2, P_3, P_4 are pairwise orthogonal projections on these eigen spaces. Also due to theorem 2,

$E^2 = \lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3 + \lambda_4 P_4$

$I = P_1 + P_2 + P_3 + P_4$

Hence expression for E^2 given above is the spectral resolution of E^2.

Since the eigen values of E^2 are 0,1,ω, and ω^2,

Spectrum of E^2 is given by

$\sigma(E^2) = \{0,1,\omega,\omega^2\}$.

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